Wave-speed dispersion associated with an attenuation obeying a frequency power law

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An attenuation scaling as a power of frequency, $|\omega|^{\beta}$, over an *infinite bandwidth* is neither analytic nor square-integrable, thus calling into question the application of the Kramers-Krönig dispersion relations for determining the frequency dependence of the associated phase speed. In this paper, three different approaches are developed, all of which return the dispersion formula for the wave-number, $K(\omega)$. The first analysis relies on the properties of generalized functions and the causality requirement that the impulse response, $k(t)$, the inverse Fourier transform of $-iK(\omega)$, must vanish for $t<0$. Second, a wave equation is introduced that yields the phase-speed dispersion associated with a frequency-power-law attenuation. Finally, it is shown that, with minor modification, the Kramers-Krönig dispersion relations with no subtractions (the Plemelj formulas) do in fact hold for an attenuation scaling as $|\omega|^{\beta}$, yielding the same dispersion formula as the other two derivations. From this dispersion formula, admissible values of the exponent β are established. Physically, the inadmissible values of β, which include all the integers, correspond to attenuation-dispersion pairs whose Fourier components cannot combine in such a way as to make the impulse response, $k(t)$, vanish for $t<0$. There is no upper or lower limit on the value that β may take.
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I. INTRODUCTION

Wave propagation in a variety of materials is characterized by an attenuation that exhibits a power-law dependence on frequency over an extended bandwidth. Such behavior is observed in underwater acoustics, medical ultrasound and geophysics, as discussed by Szabo,[1,2] who cites references from several disciplines. An attenuation that scales as a power of frequency may be expressed in the form

$$\alpha(\omega) = \alpha_0(\beta)|\omega T|^{\beta}, \tag{1}$$

where ω is angular frequency, the modulus sign ensures that $\alpha(\omega)$ is an even function of frequency, and $T = 1$ s. is a time constant with no physical significance but introduced solely for convenience in avoiding awkward units when frequency is raised to a fractional power. Both $\alpha(\omega)$ and the real, positive scaling constant α_0 have units of Nepers/unit distance; and, although it is independent of frequency, α_0 is in general a function of the exponent β, as indicated in Eq. (1).

The exponent β is a real constant, which, in practice, over a limited bandwidth, is usually observed to fall in the interval $0 < \beta \le 2$, often close to unity. In underwater acoustics, for instance, the attenuation of sound waves[3,4] and shear waves[5] in an unconsolidated marine sediment typically shows $\beta \approx 1$. Beyond the measurement bandwidth, of course, the frequency dependence of the attenuation is not known. One hypothesis is that the frequency power law in Eq. (1) is obeyed over an indefinitely broad frequency bandwidth, extending from zero to infinity. The frequency dispersion in the phase speed associated with such a power-law attenuation is the topic of this paper.

For linear propagation in an attenuating propagation medium, the wave-speed must exhibit some degree of frequency dispersion. In the case of an attenuation following a frequency power-law, the form of the associated dispersion has long been of interest[6–11] and is usually discussed in terms of the Kramers-Krönig dispersion relationships.[12,13] These integral expressions are a consequence of the causal nature of the propagating wave. In essence, causality requires that the effect, or output, cannot precede the cause, or input. In their most basic rendering, the Kramers-Krönig dispersion relations take the form of a pair of Hilbert transforms, sometimes known as the Plemelj formulas (Nussenzveig,[14] p. 24), each of which expresses the attenuation (dispersion) unambiguously as an integral of the dispersion (attenuation) over the frequency interval $(-\infty, \infty)$. Thus, in principle, the dispersion (attenuation) at a specific frequency can be derived if the attenuation (dispersion) is known at all frequencies.

Originally, the Kramers-Krönig dispersion relations were developed in connection with optics. Ginzberg,[15] some thirty years later, is generally credited with being the first to apply them to acoustics. Since then, several authors[6–8,11] have used the Kramers-Krönig dispersion relations with one or more subtractions to investigate the dispersion associated with an attenuation obeying a frequency power law of the form shown in Eq. (1). [A comprehensive discussion of subtractions in connection with the Kramers-Krönig dispersion relations has been given by Nussenzveig[14] (pp. 28–33).]

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However, despite the longstanding interest in the problem, a satisfactory expression for the dispersion has still to be developed. Horton[8] and Szabo[1] have derived a dispersion formula but, as Horton[8] himself pointed out, for certain values of the exponent β, this solution allows the phase speed to become infinite at finite frequencies, which is not physically realizable.

According to Titchmarsh's theorem (Nussenzveig,[14] p. 27), the Kramers-Krönig dispersion relationships with no subtractions, that is to say, the Plemelj formulas, hold when the reciprocal of the complex sound speed (or the refractive index in optics) characterizing the propagating wave is both square-integrable and analytic in the top half of the complex frequency plane. Neither of these conditions is satisfied by the power-law attenuation in Eq. (1), which is perhaps the source of some of the difficulties that have been encountered in previous attempts to derive the associated dispersion.

The purpose of this paper is to develop a generally valid expression for the dispersion associated with an attenuation obeying a frequency-power-law of the form shown in Eq. (1), which, unless stated otherwise, is taken to hold over an *infinitely broad band of frequency*, extending over the interval $(-\infty, \infty)$. Provided that the propagation is linear, it will be shown that such a power law is physically realizable only for certain values of the exponent β, and that these admissible values do not include the integers. In other words, an attenuation scaling precisely as $|\omega|^{n'}$ over an infinite bandwidth, where n' is any integer, cannot be exhibited by a (linear) causal signal.

Over a *limited bandwidth*, it is of course possible for the attenuation to scale as $|\omega|^{n'}$ for certain values of the integer index n', as exemplified by the solution of Stokes' wave equation[16–18] for acoustic waves in a viscous fluid. In this case, at frequencies below a transition frequency, f_T, the attenuation scales as the square of the frequency, f^2, and the phase speed is a constant, or equivalently, it varies as f^0. At higher frequencies, above the transition frequency f_T, the scaling of both the phase speed and the attenuation transitions to a square-root dependence on frequency, $f^{1/2}$ [see Eqs. (6) and (7) along with Fig. 1 in Buckingham[17]].

It is of some relevance to consider what happens when the viscosity is allowed to approach zero, representing an absence of loss in the propagation medium. In this limiting case, the behavior of the Stokes dispersion curves is quite subtle. The first point to note is that the transition frequency, f_T, scales inversely with the viscosity and hence f_T goes to infinity in the absence of loss. This represents an infinite bandwidth, $(0, \infty)$, over which the phase speed remains constant and the attenuation scales as f^2. However, the *level* of the attenuation is proportional to the viscosity, and therefore the attenuation in an inviscid fluid is zero ($0 \times f^2 = 0$).

Thus, Stokes' equation predicts that, when the phase speed is constant over an infinite bandwidth, the attenuation is identically zero, which of course is correct. (In the absence of attenuation, there is no dispersion and vice versa.) The important point here is that Stokes' equation leads to a square-law attenuation over a *finite* bandwidth (associated with non-zero viscosity) but it does not allow the same square-law scaling to extend over the *infinite* bandwidth arising from a vanishing viscosity. In terms of the power-law attenuation in Eq. (1), the dispersion curves derived from Stokes' equation prohibit the exponent β from taking the integer value 2. It will be shown more generally in the following analyses that, with an attenuation obeying a frequency power law over an infinite bandwidth, as represented by Eq. (1), all integer values of β are inadmissible.

In the following discussion, the dispersion in the phase speed associated with an attenuation in the form of a frequency power law is derived initially from a direct causality argument applied to the complex wavenumber, $K(\omega)$, of a propagating plane wave. In developing the causal approach, the only assumptions made are that the wave propagation is linear and that the Fourier transforms of the wave-speed and the attenuation exist. The latter condition holds provided that $|\omega|^\beta$ is treated as a generalized function.[19] From the solution for the dispersion, the admissible, that is to say, physically realizable values of β are established from the criterion that all the frequency components of the phase speed and the attenuation must be positive.

However, to apply this criterion, the scaling constant $\alpha_0(\beta)$ in Eq. (1) has to be specified, which raises a difficulty because $\alpha_0(\beta)$ is not returned by the constraint of causality alone. In order to determine $\alpha_0(\beta)$, a wave equation is introduced whose solution for the imaginary part of the complex wavenumber, $K(\omega)$, representing the attenuation, takes the form of a frequency power law of the type shown in Eq. (1); and the solution for the real part of $K(\omega)$ is the associated dispersion. This solution for $K(\omega)$ is valid for the exponent β any real number not an integer. The wave equation in question, originally developed in connection with shear-wave propagation in unconsolidated granular materials,[20] is based on the mechanism of strain-hardening,[21] which occurs as contiguous grains slide against one another during the passage of a wave. In effect, the inter-granular interaction becomes stiffer as the sliding progresses.

The solutions of the strain-hardening wave equation are complete, exact expressions for the frequency dependence of the phase speed and the attenuation, with the latter yielding an explicit expression for the scaling parameter $\alpha_0(\beta)$. Unlike Szabo's time domain wave equations,[2] which were derived under the assumption that the power-law attenuation is small compared with the real part of the wavenumber, no "smallness" approximation appears in the present analysis. The final expression for the wavenumber $K(\omega)$ is well-behaved at all frequencies and for all non-integer values of β, showing no singularities in the phase speed, unlike the solution of Horton[8] and Szabo.[1]

After the discussion of the wave equation, a third derivation of the phase speed associated with a power-law attenuation is developed, based on the Kramers-Krönig dispersion relations with no subtractions (i.e., the Plemelj formulas). It is shown that, in slightly modified form, these formulas hold, even though a power-law attenuation is neither analytic nor square-integrable. Moreover, the Plemelj formulas return exactly the same dispersion expression for the wavenumber $K(\omega)$ as the other two methods.

Before proceeding with the three independent derivations of the dispersion formula associated with a power-law attenuation, a linear-system argument is used to show that the complex wavenumber, $-iK(\omega)$, of a propagating plane wave is a causal Fourier transform, which is defined as *the Fourier transform of a function of time*, $k(t)$, *that is zero for all negative times*. This condition is critically important to the causality argument, for, if $-iK(\omega)$ were not a causal Fourier transform, it would not be possible to relate the phase speed to the attenuation. Having established that $-iK(\omega)$ is indeed a causal transform, the discussion continues with the development of the three derivations of the phase speed associated with a power-law attenuation.

II. THE WAVENUMBER AS A CAUSAL FOURIER TRANSFORM

Regardless of the form of the attenuation, the first step in deriving the phase speed from the attenuation is to establish that the wavenumber, $-iK(\omega)$, of a plane wave, is a causal Fourier transform. To this end, consider a plane wave, $y(t, z)$, propagating through a homogeneous medium in the positive z-direction, as illustrated in Fig. 1(a). If the source of the wave at $z = 0$ activates at time $t = 0$, then the wave at every point, $z > 0$, in the medium must satisfy the causality condition that $y(t, z) = 0$ for $t < 0$. Therefore, $Y(\omega, z)$, the temporal Fourier transform of $y(t, z)$, is a causal transform. Now, the same must be proved for the wavenumber $-iK(\omega)$.

The linear solution for the traveling wave in the frequency domain is

$$Y(\omega, z) = Y(\omega, 0)\exp[-iK(\omega)z], \tag{2}$$

where $i = \sqrt{-1}$. By considering the transmission through the thin strip between z and $z + dz$, shown in Fig. 1(a), the derivative with respect to z, denoted by a prime, may be constructed as follows:

$$\begin{aligned} Y'(\omega, z) &= \lim_{dz\to 0}\left\{\frac{Y(\omega, z+dz) - Y(\omega, z)}{dz}\right\} \\ &= -iK(\omega)Y(\omega, 0)\exp[-iK(\omega)z] \\ &= -iK(\omega)Y(\omega, z). \end{aligned} \tag{3}$$

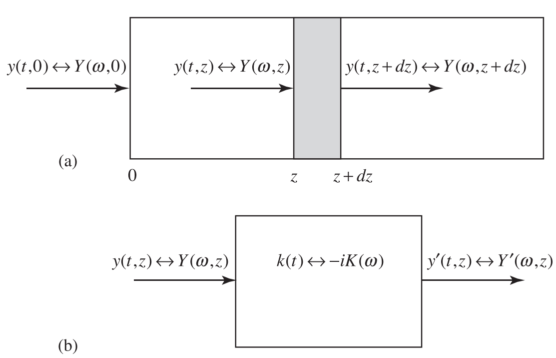

FIG. 1. (a) Transmission of a plane wave through a thin strip of a linear propagation medium located between z and $z + dz$. (b) Linear system characterized by an impulse response, $k(t)$, with input $y(t, z)$ and output the spatial derivative, $y'(t, z)$. The horizontal, double-headed arrows in both diagrams denote Fourier transform pairs.

Since it has already been shown that $Y(\omega, z)$ is a causal transform, it follows from the first of the expressions in Eq. (3) that so too is $Y'(\omega, z)$. Now, from the product in the last expression in Eq. (3), the function $Y'(\omega, z)$ may be thought of as the output of a linear system, as shown in Fig. 1(b), with a system (or transfer) function $-iK(\omega)$ and an input $Y(\omega, z)$. In the time domain, from the elementary properties of linear systems,[22] the output is the convolution

$$y'(t, z) = \int_{-\infty}^{\infty} y(\tau)k(t-\tau, z)d\tau, \tag{4}$$

where $k(t)$ is the inverse Fourier transform of the system function, $-iK(\omega)$. Setting the input equal to a Dirac delta function, $y(t) = \delta(t)$, then the output from Eq. (4) is $y'(t, z) = k(t, z)$, which proves that $k(t, z)$ is the impulse response of the linear system. Since the impulse response must be causal, vanishing for $t < 0$, it follows that the Fourier transform of $k(t)$, that is, the wavenumber $-iK(\omega)$, is indeed a causal transform.

Of course, the fact that $-iK(\omega)$ is a causal transform has long been known in acoustics,[15] and also in optics, where $K(\omega)$ is equivalent to ω times the refractive index. Most proofs, however, are predicated on the assumption that $-iK(\omega)$ is an analytic function of frequency, whereas the above argument relies solely on linearity and causality, without resorting to analyticity. This is relevant to an attenuation scaling as $|\omega|^\beta$, since the associated wavenumber, $-iK(\omega)$, is not analytic.

By writing

$$K(\omega) = \frac{\omega}{c(\omega)} - i\alpha(\omega), \tag{5}$$

it follows that, since $-iK(\omega)$ is a causal transform, the phase velocity, $c(\omega)$, must be related to the attenuation, $\alpha(\omega)$, in such a way as to ensure that the Fourier components of $-iK(\omega)$ combine to give an impulse response, $k(t)$, of precisely zero for all negative times. The nature of the relationship between phase speed and attenuation when $\alpha(\omega)$ is of the power-law form in Eq. (1) is established below. Rather than call upon the Kramers-Krönig dispersion relations, a discussion of which is deferred until Sec. VI, the following analysis is based upon the simple condition that the inverse Fourier transform of $-iK(\omega)$ must satisfy causality.

Initially, besides being real, no restriction is placed on the value that the exponent, β, may take, although non-integer and integer values of β are treated separately.

Eventually, it will be shown that, if the attenuation and dispersion are to be physically realizable, then β can take only certain non-integer values. As demonstrated later, in Sec. VI B, the Kramers-Krönig dispersion relations with no subtractions, in the form of the Plemelj formulas, are entirely consistent with the following analysis, although care must be exercised in their use when the exponent β lies outside the interval (0, 1) because then the integrals diverge. A straightforward solution to the divergence problem is introduced in Sec. VI B in the form of a continuation technique, which extends the range of convergence of the integrals involved to all real, non-integer values of β.

III. ATTENUATION, CAUSALITY, AND DISPERSION

The following analysis of the dispersion associated with a power-law attenuation relies heavily on the properties of generalized functions[19] in the evaluation of the Fourier integrals involved. The first generalized function to be introduced was the now familiar Dirac delta function, $\delta(t)$. Another is $|\omega|$, which appears in the power-law expression for the attenuation in Eq. (1). Generalized functions are not regular functions, rather, they are functionals that operate on regular, or "good" functions in such a way as to produce a defined result. (A "good" function is a function of a real variable which is everywhere differentiable any number of times. Essentially a "good" function is smoothly varying with no discontinuities. A more rigorous definition of generalized functions and "good" functions can be found in Lighthill,[19] pp. 15–17). With regard to $|\omega|^{\beta}$, it is worth noting that, since it is not analytic, certain commonly encountered theorems, such as the Paley-Wiener theorem,[1,10,23] that are predicated upon analyticity, should be treated with caution in the context of the power-law attenuation in Eq. (1).

To begin, an expression for the dispersion is developed that is generally valid, regardless of the frequency dependence of the attenuation. It is convenient to write the causal transform $-iK(\omega)$ as

$$G(\omega) = -iK(\omega) = -\alpha(\omega) - i\frac{\omega}{c(\omega)}, \tag{6}$$

where for the moment the attenuation, $\alpha(\omega)$, is taken to be quite general but will eventually be allowed to take the form shown in Eq. (1). Bearing in mind that $G(\omega)$ is the transform of a real function of time, it follows that the crossing symmetry relationship is satisfied, that is,

$$G(-\omega) = G^{*}(\omega), \tag{7}$$

where the asterisk denotes complex conjugation. Hence the first and second terms on the right of Eq. (6) must be, respectively, even and odd functions of ω, that is,

$$G(\omega) = G_{\rm even}(\omega) + iG_{\rm odd}(\omega), \tag{8a}$$

where

$$G_{\rm even}(\omega) = -\alpha(\omega), \tag{8b}$$

and

$$G_{\rm odd}(\omega) = -\frac{\omega}{c(\omega)}. \tag{8c}$$

Taking $G_{\rm even}(\omega)$, representing the attenuation, as known, the problem now is to determine $G_{\rm odd}(\omega)$, and hence the phase speed, from the causal requirement that the inverse Fourier transform of $G(\omega)$ must be zero for $t < 0$.

The inverse transform of $G(\omega)$ is

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{\rm even}(\omega)e^{i\omega t}d\omega \\ &\quad + i\frac{1}{2\pi}\int_{-\infty}^{\infty} G_{\rm odd}(\omega)e^{i\omega t}d\omega. \end{aligned} \tag{9}$$

Clearly, the last two integrals in Eq. (9) are, respectively, even and odd functions of t:

$$g(t) = g_{\rm even}(t) + g_{\rm odd}(t), \tag{10}$$

where the two real functions on the right are

$$g_{\rm even}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{\rm even}(\omega)e^{i\omega t}d\omega \tag{11a}$$

and

$$g_{\rm odd}(t) = i\frac{1}{2\pi}\int_{-\infty}^{\infty} G_{\rm odd}(\omega)e^{i\omega t}d\omega. \tag{11b}$$

To satisfy the causality condition, $g_{\rm even}(t)$ must be related to $g_{\rm odd}(t)$ in such a way as to ensure that $g(t)$ is zero for all negative times. This may be achieved by setting

$$g_{\rm odd}(t) = g_{\rm even}(t){\rm sgn}(t) + \sum_{s=0} a_{2s+1}T^{2s}\delta^{(2s+1)}(t/T), \tag{12}$$

where the discontinuous, odd function sgn(t) is a generalized function with a value of 1 for $t > 0$ and -1 for $t < 0$. The odd-order derivatives (with respect to t) of the delta-function have been included in Eq. (12) because they are all odd functions of t that are zero for $t < 0$, and thus they could in principle contribute to $g_{\rm odd}(t)$. No other such functions exist. It will turn out, however, that, when the attenuation obeys a frequency power law, the real coefficients a_{2s+1} under the summation in Eq. (12) are all zero so the summation over s could have been omitted from Eq. (12) without altering the final solution for the dispersion.

Equation (12) is the general time-domain solution for the dispersion, represented by $g_{\rm odd}(t)$, in terms of the attenuation, $g_{\rm even}(t)$. In the frequency domain, the dispersion is given by the Fourier transform

$$\begin{aligned} G_{\rm odd}(\omega) &= -\frac{|\omega|}{c(\omega)}{\rm sgn}(\omega) = -i\int_{-\infty}^{\infty} g_{\rm odd}(t)e^{-i\omega t}dt \\ &= -i\int_{-\infty}^{\infty} g_{\rm even}(t){\rm sgn}(t)e^{-i\omega t}dt \\ &\quad + i\sum_{s=0} a_{2s+1}(-i\omega T)^{2s+1}, \end{aligned} \tag{13}$$

in which the summation derives from the condition that

$$\int_{-\infty}^{\infty} \delta^{(2s+1)}(x)f(x)dx = -f^{(2s+1)}(0), \tag{14}$$

where $f(x)$ is a "good" function of time. Again, the summation over s in Eq. (13) has been included for completeness, although later it will be found to contribute nothing to the dispersion associated with an attenuation following a frequency power law.

Although the expression for the dispersion in Eq. (13) is generally valid, regardless of the functional form of the attenuation, it is particularly useful in connection with the dispersion associated with an attenuation obeying a frequency power law. In this case, the Fourier transform $G_{\text{even}}(\omega)$ may be identified with the expression for the attenuation in Eq. (1):

$$G_{\text{even}}(\omega) = -\alpha_0(\beta)|\omega T|^{\beta}. \tag{15}$$

To proceed further with the analysis, non-integer and integer values of the exponent β must be treated separately by evaluating, in both cases, the various Fourier integrals involved using the theory of generalized functions. All the Fourier transforms of generalized functions used below are listed in Table I (see the Appendix).

IV. POWER-LAW ATTENUATION (β NOT AN INTEGER)

A. Dispersion from causality

From Eq. (11a), when the exponent, β, in Eq. (1) is any real number not an integer, the inverse Fourier transform of Eq. (15) is

$$\begin{aligned} g_{\text{even}}(t) &= -(2\pi)^{-1}\alpha_0(\beta)T^{\beta}\int_{-\infty}^{\infty}|\omega|^{\beta}e^{i\omega t}d\omega \\ &= \pi^{-1}\alpha_0(\beta)T^{\beta}\sin(\beta\pi/2)\beta!|t|^{-\beta-1} \end{aligned} \tag{16}$$

and therefore from Eq. (12).

$$\begin{aligned} g_{\text{odd}}(t) &= \pi^{-1}\alpha_0(\beta)T^{\beta}\sin(\beta\pi/2)\beta!|t|^{-\beta-1}\text{sgn}(t) \\ &\quad + \sum_{s=0} a_{2s+1}T^{2s}\delta^{(2s+1)}(t/T). \end{aligned} \tag{17}$$

It follows from Eq. (13) that the dispersion, given by the Fourier transform of $g_{\text{odd}}(t)$, is

$$\begin{aligned} G_{\text{odd}}(\omega) &= -i\pi^{-1}\alpha_0(\beta)T^{\beta}\sin(\beta\pi/2)\beta!\int_{-\infty}^{\infty}|t|^{-\beta-1}\text{sgn}(t)e^{-i\omega t}dt + i\sum_{s=0}a_{2s+1}(-i\omega T)^{2s+1} \\ &= 2\pi^{-1}\sin^2(\beta\pi/2)\beta!(-\beta-1)!|\omega T|^{\beta}\text{sgn}(\omega) + i\sum_{s=0}a_{2s+1}(-i\omega T)^{2s+1} \\ &= -\alpha_0(\beta)\tan(\beta\pi/2)|\omega T|^{\beta}\text{sgn}(\omega) + i\sum_{s=0}a_{2s+1}(-i\omega T)^{2s+1}, \end{aligned} \tag{18}$$

where the identity $\beta!(-\beta-1)! = \pi\csc(\pi\beta)$, derived from the gamma function reflection formula,[24] has been used. From Eqs. (15) and (18), the expression for the complex wavenumber is

$$\begin{aligned} K(\omega) &= i[G_{\text{even}}(\omega) + iG_{\text{odd}}(\omega)] \\ &= \alpha_0(\beta)\{\tan(\beta\pi/2)\text{sgn}(\omega) - i\}|\omega T|^{\beta} \\ &\quad + \sum_{s=0}a_{2s+1}(-1)^s(\omega T)^{2s+1}. \end{aligned} \tag{19}$$

A dispersion formula similar to Eq. (19) but including an additive term linear in the frequency, corresponding to the $s = 0$ term under the summation in Eq. (19), was introduced by Horton[8] [his Eqs. (15) and (19)] and re-derived by Szabo[1] [his Eq. (18a)]. The main difference between their analyses seems to be that Horton[8] considered only positive frequencies, whereas Szabo[1] extended the treatment by formalizing it to cover all positive and negative frequencies. Neither author specified the scaling constant $\alpha_0(\beta)$ nor, it seems, recognized that it must depend on the exponent β. Horton[8] expressed concern that for certain ranges of β, the tangent function in his Eq. (15) goes negative, allowing the phase velocity to become infinite at finite frequency. Szabo,[1] curiously, seems to imply that the tangent formulation in his Eq. (18a) is satisfactory for "any reasonable value" of β.

Although they do not give an explicit expression for the complex wavenumber, Weaver and Pao[11] discuss $K(\omega)$ at length in the context of the Kramers-Krönig dispersion relations with subtractions. Rather strangely, they maintain that "… $k(t)$, the inverse Fourier transform of $K(\omega)$, is neither a causal function nor a physically meaningful function in the time domain." Actually, the impulse response, $k(t)$, is a causal function, as proved in Sec. II. Indeed, the fact that $k(t)$ is a causal function constitutes the foundation of the analysis developed above, leading to the dispersion formula in Eq. (19).

Equation (19), although exact, is not yet complete, since the factor α_0 (β) and the coefficients a_{2s+1} under the summation remain to be specified (the latter will all turn out to be zero). With regard to $\alpha_0(\beta)$, it needs to be established in order to identify the values of β that are physically realizable. Neither $\alpha_0(\beta)$ nor the a_{2s+1} coefficients, however, are available from the causality argument. Instead, it is necessary to turn to a wave equation that predicts an attenuation obeying a frequency power law.

B. Dispersion and attenuation from the strain-hardening wave equation

Just such an equation has been developed in connection with shear-wave propagation in an unconsolidated granular material.[20] The propagation medium is assumed to act as a homogeneous continuum in which internal stresses, associated with inter-granular shearing, accompany the passage of a wave. As contiguous grains slide against one another, the interaction becomes progressively stiffer, a phenomenon known as strain hardening.[21] The strain-hardening interaction is characterized by a causal material impulse response function having the form

$$h_{sh}(t) = u(t)(t/T)^{\gamma}, \tag{20}$$

where the generalized function $u(t)$ is the Heaviside unit step function, $T = 1$ s, as in Eq. (1), and γ is the material exponent, a real number sometimes known as the strain-hardening index. The subscript *sh* is a reminder that the variable is associated with the strain-hardening wave equation. Since the material is homogeneous, $h_{sh}(t)$ is independent of position in the medium.

For one-dimensional propagation in the *z*-direction, the strain-hardening wave equation is[20]

$$\frac{\partial^2}{\partial z^2}\left[h_{sh}(t) \otimes \phi(t,z)\right] - \frac{1}{c_0^2}\frac{\partial \phi(t,z)}{\partial t} = 0, \tag{21}$$

where $\phi(t, z)$ represents the propagating wave and $\otimes$ denotes a temporal convolution. The parameter c_0 is a real constant with dimensions of velocity, which may be interpreted as the phase speed that would be observed if the losses in the material were allowed to approach zero, with all else remaining the same. A temporal Fourier transform of Eq. (21), bearing in mind that the convolution in the time domain becomes a product in the frequency domain, returns the strain-hardening Helmholtz equation

$$\frac{\partial^2 \Phi(\omega,z)}{\partial z^2} + \frac{1}{c_0^2}\frac{\omega^2}{i\omega H_{sh}(\omega)}\Phi(\omega,z) = 0, \tag{22}$$

where $H_{sh}(\omega)$ is the Fourier transform of $h_{sh}(t)$:

$$\begin{aligned} H_{sh}(\omega) &= \int_{-\infty}^{\infty} h_{sh}(t) e^{-i\omega t} dt \\ &= \frac{1}{T^{\gamma}} \int_0^{\infty} t^{\gamma} e^{-i\omega t} dt. \end{aligned} \tag{23}$$

To evaluate the unilateral integral in Eq. (23), it is rewritten as the sum of two bilateral integrals:

$$\begin{aligned} H_{sh}(\omega) = &\frac{1}{2T^{\gamma}} \int_{-\infty}^{\infty} |t|^{\gamma} e^{-i\omega t} dt \\ &+ \frac{1}{2T^{\gamma}} \int_{-\infty}^{\infty} |t|^{\gamma} \mathrm{sgn}(t) e^{-i\omega t} dt. \end{aligned} \tag{24}$$

Appealing once again to the theory of generalized functions, these two integrals may be evaluated with the aid of the Fourier transforms in Table I to yield

$$\begin{aligned} H_{sh}(\omega) = &\gamma! T [\cos\{(\gamma+1)\pi/2\} \\ &- i\,\mathrm{sgn}(\omega)\sin\{(\gamma+1)\pi/2\}] |\omega T|^{-\gamma-1}, \end{aligned} \tag{25}$$

which simplifies to

$$H_{sh}(\omega) = \gamma! T (i\omega T)^{-\gamma-1}. \tag{26}$$

Equations (25) and (26) are valid for γ any real number not an integer.

According to Eq. (22),

$$\frac{1}{c_{sh}(\omega)} - i\frac{\alpha_{sh}(\omega)}{\omega} = \frac{1}{c_0}\left[i\omega H_{sh}(\omega)\right]^{-1/2}, \tag{27}$$

where c_{sh} and α_{sh} are, respectively, the phase speed and attenuation of the propagating wave at angular frequency ω. It is readily shown from Eqs. (26) and (27) that, with γ any real number not an integer, the phase speed is given by

$$\frac{1}{c_{sh}(\omega)} = \frac{1}{c_0}(\gamma!)^{-1/2}|\omega T|^{\gamma/2}\cos(\gamma\pi/4) \tag{28a}$$

and the attenuation is

$$\alpha_{sh}(\omega) = -\frac{1}{c_0 T}(\gamma!)^{-1/2}|\omega T|^{1+\gamma/2}\sin(\gamma\pi/4). \tag{28b}$$

The expressions in Eq. (28) specify the phase speed and attenuation completely, including the previously unknown scaling parameter $\alpha_0(\beta)$. As a check on Eq. (28), let γ approach zero, representing an absence of strain-hardening, in which case the attenuation goes to zero and the phase speed reduces to the constant c_0. This is clearly correct and is consistent with the behavior predicted by Stokes' equation when the viscosity is allowed to approach zero.

C. The scaling constant $\alpha_0(\beta)$

It is evident from Eq. (28b) that the attenuation predicted by the strain-hardening wave equation does indeed take the form of a frequency power law extending from zero up to indefinitely high frequencies. No other linear wave equation is known to exist that has such a solution. A comparison of Eq. (28b) with Eq. (1) shows that the exponents β and γ are related through the simple expression

$$\beta = 1 + \frac{\gamma}{2}, \tag{29}$$

and that the scaling constant, $\alpha_0(\beta)$, is

$$\begin{aligned} \alpha_0(\beta) &= -\frac{1}{c_0 T}(\gamma!)^{-1/2}\sin(\gamma\pi/4) \\ &= \frac{1}{c_0 T}\{[2(\beta-1)]!\}^{-1/2}\cos(\beta\pi/2). \end{aligned} \tag{30}$$

Equation (30) is the required solution for $\alpha_0(\beta)$ that cannot be determined from causality considerations alone.

Now that $\alpha_0(\beta)$ has been identified, the complete expression for the wavenumber, $K(\omega)$, may be obtained from Eq. (19), as derived from the causality argument, or from the solution of the strain-hardening wave equation in Eq. (28). Either way, the wavenumber is found to be

$$\begin{aligned} K(\omega) = &\frac{1}{c_0 T}\{[2(\beta-1)]!\}^{-1/2} \\ &\times\left[\mathrm{sgn}(\omega)\sin(\beta\pi/2) - i\cos(\beta\pi/2)\right]|\omega T|^{\beta}, \end{aligned} \tag{31}$$

where the summation over the index s in Eq. (19) has been omitted since its absence from the solution of the strain-hardening wave equation in Eq. (28) signifies that all the a_{2s+1} coefficients are zero.

The dispersion formula in Eq. (31) is the general solution, valid for β any real number not an integer, for the wavenumber associated with a frequency-power-law attenuation. It is interesting to note that, according to Eq. (31), the real and imaginary parts of the wavenumber, representing the dispersion and attenuation, respectively, exhibit exactly the same functional dependence on frequency: both scale as $|\omega|^{\beta}$.

Such behavior is not exhibited by the formulation of Horton[8] [his Eq. (19)] and Szabo[1] [his Eq. (18a)]. In fact, their solution differs from Eq. (31) in a number of ways, as may be appreciated by writing the Horton/Szabo expression for the wavenumber as follows:

$$K_{HS}(\omega) = \frac{\omega}{c(\omega_1)} + \alpha_0\left[\mathrm{sgn}(\omega)\tan(\beta\pi/2) - i\right]|\omega T|^{\beta}, \quad (32)$$

where $c(\omega_1)$ is the phase speed at some fixed frequency, ω_1, and α_0 is the same scaling constant as in Eq. (1) but treated by Horton[8] and Szabo[1] as independent of the exponent β. Horton[8] derived Eq. (32) from his Eq. (1), which, in effect, is the Kramers-Krönig dispersion relation with one subtraction. The subtraction gives rise to the first term on the right of Eq. (32), which is not present in Eq. (31). This additive term is problematical on a number of levels, not least because there is no preferred choice for the frequency ω_1 when the attenuation is in the form of a power law. Horton[8] takes ω_1 to be zero, whereas Szabo[1] argues, for reasons that are not entirely clear, that ω_1 should be infinity for $0 < \beta < 1$ and zero for $1 < \beta < 2$. In the context of a power-law attenuation, no subtractions need be applied to the Kramers-Krönig relations. It will be shown later, in Sec. VI, that the Kramers-Krönig relations with no subtraction, otherwise known as the Plemelj formulas, yield exactly the same expression for the wavenumber as shown in Eq. (31).

Another dissimilarity to note is that the tangent function in the real part and the constant in the imaginary part of the second term on the right of Eq. (32) have been replaced in Eq. (31) by sine and cosine functions, respectively. The fact that the tangent function can go negative, thus allowing the real part of the expression on the right of Eq. (32) to go to zero, was the source of Horton's[8] concern that the phase speed predicted by Eq. (32) can become infinite at finite frequencies. Such pathological behavior may be attributed to the presence of the additive phase speed-term on the right of Eq. (32). Such non-physical behavior is not exhibited by the formulation in Eq. (31), where there is no additive term, and the sine function is always non-zero, since β in this expression is constrained to be non-integer, and hence the phase speed is always finite.

Besides leading to an infinite phase speed at finite frequencies, the additive phase-speed term on the right of Eq. (32) gives rise to another non-physical characteristic. Whenever β is an even number, $\beta = 2n'$, where n' is any integer, the tangent function is identically zero, in which case Eq. (32) returns a constant phase speed. The associated attenuation takes the form of a frequency power law, $\omega^{2n'}$. But this is impossible. It is an elementary fact that the phase speed and the attenuation are uniquely related and that, when the phase speed is constant, the attenuation is zero (as discussed in Sec. I in connection with Stokes' wave equation). Therefore, a constant phase speed cannot also be associated with an attenuation following a frequency power law, let alone with an infinite number of power laws, each corresponding to one of the integers, $n' = \ldots, -2, -1, 0, 1, 2, \ldots$. No such problem is encountered with Eq. (31), in which β is constrained to be non-integer. Integer values of β are discussed below and shown to be physically unrealizable.

A further feature that distinguishes Eq. (31) from Eq. (32) is the presence of the factorial function in Eq. (31), which must be greater than zero, otherwise, as may easily be verified, the phase speed and the attenuation would both be odd functions of frequency, which violates the crossing symmetry condition in Eq. (7) and is therefore not permissible. This places a restriction on the possible values that can be taken by β. The requirement that the phase speed and attenuation should both be even functions of frequency, corresponding to the factorial function $[2(\beta - 1)]!$ being positive, is referred to below as the *first existence condition*.

D. Admissible values of β

Additional constraints on the values that β may take can be established directly from Eq. (31), since it specifies fully the β-dependence of the wavenumber $K(\omega)$. For the propagating wave to be physically realizable, a necessary requirement, referred to hereafter as the *second existence condition*, is that the phase speed and the attenuation should both be positive at all frequencies. Otherwise, waves could travel back toward the source (negative phase speed) or increase in amplitude with increasing range from the source (negative attenuation), neither of which is physically possible. Other physically unrealizable conditions are a phase speed or attenuation of zero, and a phase speed or attenuation of infinity at finite frequencies. (Relativistic and quantum mechanical effects are not considered here.) Values of β that satisfy both the first and second existence conditions correspond to phase speed and attenuation pairs whose frequency components can combine in such a way as to ensure that $-iK(\omega)$ is not only a causal transform but is also physically realizable.

To examine the implications of the first existence condition, the factorial function in Eq. (31) is expressed in terms of the gamma function:

$$[2(\beta - 1)]! = \Gamma(2\beta - 1), \quad (33)$$

which is plotted in Fig. 2 for positive and negative ranges of the argument. Throughout the positive range of the argument, corresponding to $\beta > 0.5$, the gamma function is positive, and thus the first existence condition is always satisfied.

Over the negative range of the argument, corresponding to $\beta < 0.5$, the gamma function is positive, satisfying the first existence condition, when $-(n + 0.5) \le \beta < -n$, where $n \ge 0$ is any non-negative integer. In contrast, the gamma function is negative in the negative range of the argument when $-n < \beta \le -(n - 0.5)$, and therefore such values of β are

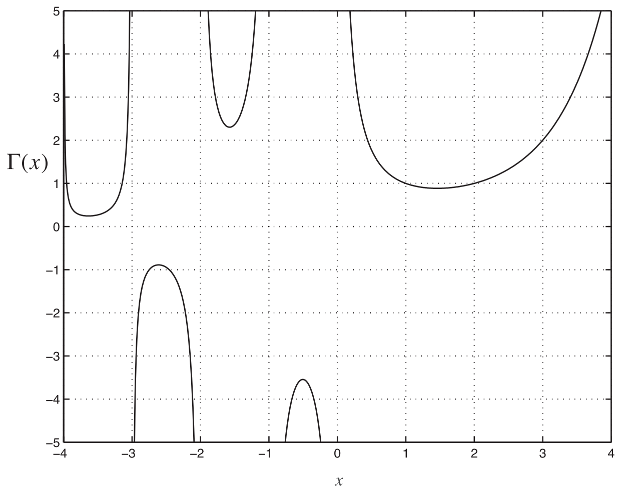

FIG. 2. The gamma function, $\Gamma(x)$, for real argument, x.

prohibited, since they fail to satisfy the first existence condition. In particular, when $n = 0$, corresponding to $0 < \beta \leq 0.5$, the first existence condition is not satisfied and hence an attenuation obeying a frequency power law with an exponent, β, in the interval (0, 0.5) could not exist in practice.

Turning now to the second existence condition, the sine function in Eq. (31) is positive when $4n' < \beta < 4n' + 2$, and the cosine when $4n' - 1 < \beta < 4n' + 1$, where n' is any integer. Both the sine and the cosine are positive when $4n' < \beta < 4n' + 1$, which identifies the ranges of β allowed by the second existence condition.

The overlap ranges of β allowed by the first $[-(n + 0.5) \leq \beta < -n$ and $\beta > 0.5]$ and second $[4n' < \beta < 4n' + 1]$ existence conditions represent the non-integer values of β for which an attenuation following a frequency power-law is physically realizable. For $\beta < 1$, these overlap ranges representing the admissible values of β are given by

$$-4n + 0.5 \leq \beta < -4n + 1 \tag{34a}$$

and, for $\beta > 1$,

$$4n + 4 < \beta < 4n + 5, \tag{34b}$$

where, as before, n is any non-negative integer. For example, the allowed values of β that are clustered around $\beta = 0$ fall in the intervals (−7.5, −7), (−3.5, −3), (0.5, 1), (4, 5), (8, 9).

Although these intervals are sparsely distributed, it is nevertheless evident that in principle β may take values, between four and five for example, that are greater than unity. Obviously in such cases, in the limit of high frequency, the real and imaginary parts of the wavenumber, $K(\omega)$, rise faster than the first power of frequency. This observation is in contradiction with the conclusion reached by Weaver and Pao,[11] from considerations involving the Kramers-Krönig dispersion relations with two subtractions, that the attenuation, as $\omega \to \infty$, cannot rise faster than the first power of frequency. In fact, there is no upper or lower limit on the value that β may take. Equation (34) is also in disagreement with Szabo's[1] conclusion that the dispersion is cyclic due to the periodic nature of the tangent function in Eq. (32). The term in square brackets containing the trigonometric functions in Eq. (31) is indeed periodic but, as may be seen in Eq. (34a) and Eq. (34b), the cyclical constraints it imposes on the permissible values of β are very different from those related to the tangent function in Eq. (32).

It is interesting to note that, according to the dispersion formula in Eq. (31), β is prohibited in the interval (1, 2). In practice, values of β falling within this inadmissible range have been observed[1] over *limited* bandwidths. For instance, in silicone fluid,[25] from 2.25 to 10 MHz, $\beta = 1.79$; and in castor oil,[26] from approximately 0.5 to 50 MHz, $\beta = 1.66$. It should be appreciated that such behavior, observed over a bandwidth of two decades or less, is not incompatible with the dispersion formula in Eq. (31), which, it will be recalled, relates to an attenuation scaling as $|\omega|^\beta$ over an *infinite* bandwidth, not just a limited range of frequency.

V. POWER-LAW ATTENUATION (β AN INTEGER)

A. Dispersion from causality

When the exponent β is an integer, the dispersion may be derived from the attenuation based on a causality argument analogous to that in Sec. IV A. As before, the starting point is the inverse Fourier transform of the function in Eq. (15):

$$g_{\text{even}}(t) = -(2\pi)^{-1}\alpha_0(\beta)T^\beta \int_{-\infty}^{\infty} |\omega|^\beta e^{i\omega t} d\omega. \tag{35}$$

Now, however, because of the nature of the Fourier transforms of the generalized functions involved, it is necessary to treat positive and negative values of β separately.

First, consider the case where $\beta = 2n + 1$ is a positive odd integer. From Table I, Eq. (35) then evaluates to

$$g_{\text{even}}(t) = (-1)^n \pi^{-1}\alpha_0(2n+1)T^{2n+1}(2n+1)!\,t^{-2n-2}, \quad n = 0, 1, 2, \ldots, \tag{36a}$$

and the dispersion, from Eq. (13), is

$$\frac{\omega}{c(\omega)} = -G_{\text{odd}}(\omega) = -2\pi^{-1}\alpha_0(2n+1)(\omega T)^{2n+1} \times \ln(|\omega|/\omega_0), \ n = 0, 1, 2, \ldots, \tag{36b}$$

where ω_0 in the argument of the logarithm is an unspecified positive constant. For $\beta = 2n$, a non-negative even integer,

$$g_{\text{even}}(t) = (-1)^{n+1}(2\pi)^{2n-1}\alpha_0(2n)T^{2n}\delta^{(2n)}(t), \quad n = 0, 1, 2, \ldots, \tag{37a}$$

and the dispersion, from Eq. (13), is

$$\frac{\omega}{c(\omega)} = -G_{\text{odd}}(\omega) = i(-1)^{n+1}(2\pi)^{2n-1}\alpha_0(2n)T^{2n} \times \int_{-\infty}^{\infty} \delta^{(2n)}(t)\,\text{sgn}(t)e^{-i\omega t}dt, \ n = 0, 1, 2, \ldots. \tag{37b}$$

The summation over odd powers of frequency in Eq. (13) has been neglected in Eqs. (36b) and (37b) because it makes no difference to the conclusion reached below, namely, that neither of these expressions is physically realizable.

According to Eq. (36b), the dispersion is logarithmic when the attenuation scales as a positive, odd power of frequency. Indeed, when β is unity (i.e., $n=0$), representing an attenuation that is proportional to frequency, the form of the dispersion in Eq. (36b) is reminiscent of Futterman's logarithmic dependence on frequency[6] [his Eq. (25)]. At high frequencies, when $|\omega| > \omega_0$, the expression for the dispersion in Eq. (36b) goes negative and in so doing violates the second existence condition. Futterman[6] recognized the problem and avoided it by truncating the power-law attenuation. The unavoidable conclusion is that an attenuation obeying a frequency power law over an *infinite bandwidth* with the exponent β a positive, odd integer is not physically realizable.

The expression for the dispersion in Eq. (37b), for β a non-negative even integer is also non-physical, in this case because the integral is indeterminate: the integrand contains the product of two generalized functions (the derivative of the delta function times the signum function), which is not defined. It follows from Eq. (37b) that an attenuation obeying an *infinite-bandwidth* frequency power law in which the exponent, β, is an even non-negative integer cannot exist.

Turning now to the non-positive integers, when the exponent is odd such that $\beta=-(2n+1)$, where, as earlier, $n\geq 0$ is any non-negative integer, then the inverse Fourier transform in Eq. (35) evaluates to

$$g_{\rm even}(t) = \frac{(-1)^n}{\pi(2\pi)!}\alpha_0(-2n-1)t^{2n}\{\ln|t|+C\},\quad n=0,1,2,\ldots \tag{38a}$$

where C is an unspecified constant. After substituting Eq. (38a) into Eq. (13), the dispersion is found to be

$$\frac{\omega}{c(\omega)} = -G_{\rm odd}(\omega) = -2\pi^{-1}\alpha_0(-2n-1)(\omega T)^{-2n-1} \times\{\ln(|\omega| - \psi(n) - C\},\ n=0,1,2,\ldots, \tag{38b}$$

where

$$\psi(x) = \frac{d}{dx}\ln x!. \tag{39}$$

Again, the summation over odd powers of frequency in Eq. (13) has been neglected, since it does not alter the following conclusion. At sufficiently high frequencies the logarithmic term in Eq. (38b) leads to negative phase speeds, in violation of the second existence condition, which forces the conclusion that odd, negative values of the exponent β are non-physical and cannot be realized in practice.

When $\beta=-2\,m$ is an even, negative integer, then from Eq. (34),

$$g_{\rm even}(t) = \frac{(-1)^m}{2(2m-1)!}\alpha_0(-2m)T^{-2m}t^{2m-1}{\rm sgn}(t),\quad m=1,2,\ldots \tag{40a}$$

and from Eq. (13)

$$\frac{\omega}{c(\omega)} = -G_{\rm odd}(\omega) = \frac{(2\pi)^{2m-1}}{2(2m-1)!}\alpha_0(-2m)T^{-2m}\delta^{(2m-1)}(\omega),\ m=1,2,\ldots, \tag{40b}$$

where, again, the summation over s has been neglected, since it does not alter the following conclusion. According to Eq. (40b), the phase speed is the reciprocal of the derivative of a delta function, which is indeterminate. Thus, an attenuation exhibiting a frequency power law in which the exponent is an even negative integer is not physically realizable.

In summary, the above arguments show that it is not possible for the exponent β to take any integer value (negative, positive or zero). If the attenuation were to obey a frequency power law with an integer exponent, the associated phase speed would be, in one way or another, non-physical. It follows that such an attenuation, extending over the indefinitely wide frequency interval $(0, \infty)$, could not be encountered in practice.

B. Limiting behavior

It is evident from the above discussion that integer values of β lead to ill-conditioned dispersion behavior, the character of which is fundamentally different from the dispersion associated with non-integer β. Some insight as to why this might be may be gained by considering the limiting forms of the dispersion and attenuation as non-integer β is allowed to approach integer values. This may be facilitated from the dispersion formula in Eq. (31), which is valid for β any real number not an integer, by expressing the dispersion and attenuation explicitly as

$$\frac{|\omega|}{c(\omega)} = \frac{1}{c_0 T\{\Gamma(2\beta-1)\}^{1/2}}\sin(\beta\pi/2)|\omega T|^{\beta} \tag{41a}$$

and

$$\alpha(\omega) = \frac{1}{c_0 T\{\Gamma(2\beta-1)\}^{1/2}}\cos(\beta\pi/2)|\omega T|^{\beta}. \tag{41b}$$

In passing, it is worth noting the symmetry between the dispersion and the attenuation in Eq. (41), a symmetry that is absent from the Horton/Szabo formulation in Eq. (32).

First, suppose that β approaches any even, positive integer infinitesimally closely, then the sine function in Eq. (41a) approaches zero, in which case the phase speed diverges to infinity. Thus, even, positive integer values of β are inadmissible because an infinite phase speed at finite frequencies is non-physical. When β is any odd, positive integer, the cosine function in Eq. (41b) approaches zero, corresponding to an attenuation of zero, which again is non-physical, since all materials must exhibit some degree of loss, however small. It is interesting to note, however, that, according to Eq. (40), as beta approaches unity from below, the phase speed becomes constant and the attenuation goes to zero, which is correct.

Now, let β approach infinitesimally closely to any negative integer or zero. In this case, the gamma function in the denominators of Eqs. (41) diverges to infinity (see Fig. 2), giving rise to a wave-speed of infinity and an attenuation of zero. Again, such non-physical behavior could not be realized in practice.

In general, according to the limiting forms of Eq. (41), any integer value of β gives rise to a phase speed of infinity or an attenuation of zero, neither of which is physically realizable. It is not, therefore, surprising that the earlier causality argument in Sec. V A returns non-physical or indeterminate results for the dispersion when β is an integer. Neither are the Kramers-Krönig dispersion relations (the Plemelj formulas), which have yet to be discussed, immune from problems when β is an integer, in this case because the integrals fail to converge. As demonstrated below, however, the Plemelj formulas (with a minor modification) are well behaved and consistent with the dispersion formula in Eq. (31) when β is any real number not an integer.

VI. KRAMERS-KRÖNIG RELATIONS

Most of the analyses[1,6,8,11,15,27] of dispersion associated with an attenuation obeying a frequency power law that have appeared in the literature are based on the Kramers-Krönig dispersion relations with one or more subtractions. Horton,[8] for instance, rests his whole argument on the Kramers-Krönig dispersion relation with one subtraction [his Eq. (1)], which includes an additive constant in the form of the phase speed at some unidentified (and unidentifiable) frequency. Weaver and Pao,[11] struggling with the fact that a power-law attenuation diverges to infinity at infinite frequency, use two subtractions [their Eqs. (71) and (72)], again involving unknown and difficult to specify constants.

Although causality is fundamental to the derivation of the expression for the wavenumber in Eq. (31), the Kramers-Krönig dispersion relations have not featured in the analysis of the dispersion presented above. However, it is possible to derive the dispersion formula in Eq. (31) from the basic Kramers-Krönig dispersion relations with no subtractions, otherwise known as the Plemelj formulas, as demonstrated below.

On a cautionary note, before proceeding, it is worth mentioning that certain confusing statements have appeared in the literature in connection with the Kramers-Krönig dispersion relations and their application to the problem of dispersion associated with an attenuation scaling as $|\omega|^{\beta}$. Weaver and Pao in Ref. 11, p. 1910, for example, say that the "…connection between causality and analyticity is at the root of all dispersion relations." An attenuation scaling as $|\omega|^{\beta}$, however, is not analytic, yet it gives rise to the dispersion formula in Eq. (31), which, as has been shown, can be derived from an argument based on the requirement that causality should be satisfied.

The function $|\omega|^{\beta}$ is not only non-analytic, it is not square-integrable, or even bounded. Yet, square integrability and analyticity are basic requirements of Titchmarsh's theorem (Nussenzveig,[14] p. 27), which states that a causal Fourier transform will satisfy the Kramers-Krönig dispersion relations in the form of the Plemelj formulas, and conversely that if the Plemelj formulas are satisfied, the function in question is a causal transform. Since $|\omega|^{\beta}$ is not square integrable, the question naturally arises as to whether Titchmarsh's theorem holds and the Plemelj formulas are valid for an attenuation obeying a frequency-power-law? Clearly, the usual derivation of the Plemelj formulas, which relies on square-integrability, analyticity and contour integration in the complex frequency plane, is not appropriate for an attenuation scaling as the non-analytic function $|\omega|^{\beta}$.

It is shown below that the Plemelj formulas are in fact valid when the attenuation varies as $|\omega|^{\beta}$, although, because of convergence issues, they do not necessarily hold for the function $-iK(\omega)$, as discussed later in Sec. VI B. Beforehand, a straightforward derivation of the Kramers-Krönig dispersion relations with no subtractions (the Plemelj formulas) is developed, based on Fourier integrations taken along the real frequency axis, without resorting to analytic continuation into the complex plane. There is no requirement that the causal transform in question be square-integrable or analytic, only that the relevant Fourier transforms exist.

A. Derivation of the Kramers-Krönig relations (the Plemelj formulas)

A real, causal function of time,

$$q(t) = 0 \quad \text{for } t < 0, \tag{42}$$

may be expressed in terms of its Fourier transform, $Q(\omega)$, as

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega')e^{i\omega' t}d\omega'. \tag{43}$$

The Fourier transform itself is

$$Q(\omega) = \int_{0}^{\infty} q(t)e^{-i\omega t}dt, \tag{44}$$

where the lower limit of zero is a consequence of the causality condition in Eq. (42). On substituting the expression for $q(t)$ in Eq. (43) into the integrand of Eq. (44), the Fourier transform becomes the double integral

$$Q(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} e^{i(\omega'-\omega)t}dt\int_{-\infty}^{\infty} Q(\omega')d\omega', \tag{45}$$

where, from the theory of generalized functions, the integral over time is

$$\int_{0}^{\infty} e^{i(\omega'-\omega)t}dt = \pi\delta(\omega'-\omega) + \frac{i}{(\omega'-\omega)}. \tag{46}$$

It follows that Eq. (45) reduces to

$$Q(\omega) = \frac{i}{\pi}P\int_{-\infty}^{\infty} \frac{Q(\omega')}{(\omega'-\omega)}d\omega', \tag{47}$$

where the P before the integral denotes a Cauchy principal value. Thus, according to Eq. (47), the causal transform $Q(\omega)$ may be expressed as a Hilbert transform of itself.

On writing $Q(\omega)$ in terms of its real and imaginary parts,

$$Q(\omega) = R(\omega) + iX(\omega) \tag{48}$$

then, by equating the real parts of Eq. (47), it follows that

$$R(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{X(\omega')}{(\omega' - \omega)} d\omega' \tag{49a}$$

and, by equating the imaginary parts, that

$$X(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{R(\omega')}{(\omega' - \omega)} d\omega'. \tag{49b}$$

The Hilbert transforms in Eq. (49) are the well known Kramers-Krönig dispersion relations (with no subtractions), referred to by Nussenzveig[14] (p. 24) as the Plemelj formulas. Since $q(t)$ is a real function of time, the crossing symmetry condition $Q(\omega) = Q^*(-\omega)$ must be satisfied, allowing the bilateral integrals in Eq. (49) to be expressed in the alternative, unilateral form as follows:

$$R(\omega) = -\frac{2}{\pi} P \int_{o}^{\infty} \frac{\omega' X(\omega')}{(\omega'^2 - \omega^2)} d\omega' \tag{50a}$$

and

$$X(\omega) = \frac{2\omega}{\pi} P \int_{-\infty}^{\infty} \frac{R(\omega')}{(\omega'^2 - \omega^2)} d\omega'. \tag{50b}$$

The expressions in Eq. (49) and (50) are standard results that are usually derived from contour integration in the complex frequency plane on the assumption that $Q(\omega)$ is square-integrable and analytic. Since no such assumptions have been made here, Eq. (49) and Eq. (50) are applicable to the case of an attenuation obeying a frequency power law of the type shown in Eq. (1). Below, the bilateral form in Eq. (49a) is used to derive the phase speed associated with a power-law attenuation.

B. Frequency power law attenuation (β a real number not an integer)

A central feature of the following discussion is the idea that dividing or multiplying a causal transform by $i\omega$ yields another causal transform. This may be appreciated from the fact that, in the time domain, such operations correspond to integration and differentiation, respectively. To begin with a simple example, suppose that the causal transform $Q(\omega)$ in Eq. (48) is identified, not as $-iK(\omega)$, but as $-iK(\omega)/(i\omega)$, then the imaginary part, $X(\omega)$, is equal to

$$X(\omega) = \alpha(\omega)/\omega = \alpha_0(\beta)T^{\beta}|\omega|^{\beta-1}\mathrm{sgn}(\omega). \tag{51a}$$

From the first Kramers-Krönig integral, Eq. (49a), the corresponding real part representing the dispersion is

$$R(\omega) = -\frac{\alpha_0(\beta)T^{\beta}}{\pi} P \int_{-\infty}^{\infty} \frac{|\omega'|^{\beta-1}}{\omega' - \omega} \mathrm{sgn}(\omega') d\omega'. \tag{51b}$$

Now, the integral here is a known form that can be found in Gradshteyn and Ryzhik,[28] which yields

$$R(\omega) = -\alpha_0(\beta)T^{\beta}|\omega|^{\beta-1}\tan(\beta\pi/2), \text{ for } 0 < \beta < 1. \tag{52}$$

By combining the expressions in Eq. (52) and Eq. (51a), it is easily demonstrated that the tangent formulation in Eq. (19) for the wavenumber, $K(\omega)$, is recovered identically. The summation over odd powers of frequency, however, is absent from the Kramers-Krönig solution, signifying that the coefficients a_{2s+1} are all zero, which is consistent with the dispersion formula in Eq. (31), as derived from the strain-hardening wave equation.

It is important to note, however, that the integral in Eq. (51b) converges only for β in the interval (0, 1). To extend the validity of the Plemelj formulas to all real, non-integer values of β, the exponent is written as

$$\beta = n' + \beta', \tag{53}$$

where n' is the value of β rounded down to the nearest integer, which may be positive, negative or zero, and $0 < \beta' < 1$ is the remainder. Now the causal transform $Q(\omega)$ in Eq. (48) is identified, not as $-iK(\omega)/(i\omega)$, but more generally as

$$Q(\omega) = -\frac{iK(\omega)}{(i\omega)^{n'+1}} = -\frac{1}{i^{n'}}\left\{\frac{1}{\omega^{n'}c(\omega)} - i\frac{\alpha(\omega)}{\omega^{n'+1}}\right\}. \tag{54}$$

As demonstrated below, the expression in Eq. (54) satisfies the Plemelj formulas for n' any integer. Obviously, in the special case when $n' = 0$, the exponent β falls in the interval (0, 1) and $Q(\omega)$ in Eq. (54) reverts to $-iK(\omega)/(i\omega)$, in which case, as already shown, the Kramers-Krönig integral in Eq. (49a) returns the tangent formula in Eq. (52).

Of greater interest is the general case of n' any integer. Since the factor $1/i^{n'}$ in Eq. (54) is real when n' is even and imaginary when n' is odd, it is necessary to treat even and odd values of n' separately. First, taking n' to be even, then $b = -1/i^{n'}$ is real, thus the real and imaginary parts of $Q(\omega)$ are, respectively,

$$R(\omega) = \frac{b}{\omega^{n'}c(\omega)} \tag{55a}$$

and

$$X(\omega) = -b\frac{\alpha(\omega)}{\omega^{n'+1}} = -b\alpha_0(\beta)T^{\beta}|\omega|^{\beta'-1}\mathrm{sgn}(\omega). \tag{55b}$$

$R(\omega)$ may be derived from the first of the Plemelj formulas in Eq. (49a), which yields

$$R(\omega) = \frac{b}{\pi}\alpha_0(\beta)T^{\beta} P \int_{-\infty}^{\infty} \frac{|\omega'|^{\beta'-1}}{\omega' - \omega}\mathrm{sgn}(\omega')d\omega' = b\alpha_0(\beta)T^{\beta}|\omega|^{\beta'-1}\tan(\beta'\pi/2), \tag{56}$$

where the integral, the same as that in Eq. (51b), can be found in Gradshteyn and Ryzhik.[28] The wavenumber may now be written as

$$K(\omega) = b^{-1}\omega^{n'+1}\{R(\omega) + iX(\omega)\}$$
$$= \alpha_0(\beta)[\mathrm{sgn}(\omega)\tan(\beta\pi/2) - i]|\omega T|^{\beta}, \tag{57}$$

where $\beta' = \beta - n'$ has been used to obtain the argument of the tangent function. A similar derivation, but with n' odd, leads to exactly same expression for the wavenumber as that in Eq. (57), which is therefore valid for all real, non-integer β.

When the solution in Eq. (30) for the scaling constant $\alpha_0(\beta)$ is substituted into Eq. (57), the expression for the wavenumber becomes

$$K(\omega) = \frac{1}{c_0 T}\{[2(\beta-1)]!\}^{-1/2} \times \left[\mathrm{sgn}(\omega)\sin(\beta\pi/2) - i\cos(\beta\pi/2)\right]|\omega T|^{\beta}, \tag{58}$$

which is identical to the dispersion formula in Eq. (31), as derived from both the causality argument and the strain-hardening wave equation. Since no subtractions were used in arriving at Eq. (57), the Kramers-Krönig analysis developed above has introduced no spurious, awkward to specify constants into the solution for the wavenumber. This is equivalent to saying that, according to the Plemelj formulas, the coefficients a_{2s+1} under the summation sign in Eq. (19) are all zero.

In summary, the key to the use of the Kramers-Krönig dispersion relations (Plemelj formulas) in the context of a power-law attenuation is to operate, not with $-iK(\omega)$, but with $-iK(\omega)/(i\omega)^{n'+1}$, where n' is the value of β rounded down to the nearest integer. The reduced exponent is then the remainder, β', which always falls in the interval (0, 1), and thus the Hilbert transforms constituting the Kramers-Krönig dispersion relations converge, taking forms that are readily available from tables of integrals.

C. Frequency power law attenuation (β an integer)

When $\beta = n'$, where n' is any integer, the Hilbert transforms in Eq. (49) fail to converge. Moreover, convergence cannot be achieved by using a procedure analogous to that described above, whereby the exponent β is reduced to a value lying between zero and unity. The fact that the Kramers-Krönig integrals in Eq. (49) diverge when β is an integer indicates that, in this situation, the phase speed is not causally connected to the attenuation. This is consistent with the earlier discussions of the impulse response and the strain-hardening wave equation, in both of which, integer values of β were found to be non-realizable. Physically, this means that, with $\beta = n'$, the frequency components in the phase speed and attenuation cannot combine in such a way as to make the impulse response, $k(t)$, zero for $t < 0$.

D. Titchmarsh's theorem

Consider the function $Q(\omega)$ in Eq. (48) and suppose for the moment that it is square-integrable, so it satisfies

$$\int_{-\infty}^{\infty} |Q(\omega)|^2 d\omega < C, \tag{59}$$

where C is a finite constant. Titchmarsh's theorem (Nussenzveig,[14] p. 27) states that if $Q(\omega)$ is square-integrable and it fulfills one of the following three conditions, it fulfills all three of them: (1) the inverse Fourier transform, $q(t)$, vanishes for $t < 0$; (2) the first Plemelj formula in Eq. (49a) holds; and (3) the second Plemelj formula in Eq. (49b) holds. Actually, a fourth condition is also included in the theorem, involving analytic continuation into the complex ω-plane, which is not relevant to the present discussion.

Now suppose that $Q(\omega) = -iK(\omega)$, where $K(\omega)$ is the wavenumber in Eq. (58) associated with an attenuation obeying a frequency power law. In this case, since the condition for square-integrability in Eq. (59) does not hold, $Q(\omega)$ fails to fulfill the essential requirement upon which Titchmarsh's theorem is predicated. Nevertheless, $Q(\omega)$ is a causal transform for all non-integer β, that is, its inverse transform, $q(t)$ vanishes for $t < 0$, thereby fulfilling the first Titchmarsh condition. However, the second and third Titchmarsh conditions, namely, the Plemelj formulas in Eq. (49), are fulfilled by $Q(\omega)$, but only if β lies in the interval (0, 1). Otherwise, for values of β falling outside this interval, the Plemelj formulas do not converge. Still, even though it is not square integrable, $Q(\omega)$ conforms to Titchmarsh's theorem provided β is restricted to values in the interval (0, 1).

Moving on to the more general case, suppose that $Q(\omega)$ takes the form in Eq. (54):

$$Q(\omega) = -\frac{iK(\omega)}{(i\omega)^{n'+1}}, \tag{60}$$

where $K(\omega)$ is the wavenumber in Eq. (58) and n' is still the value of β rounded down to the nearest integer. Again, $Q(\omega)$ is not square-integrable since it does not satisfy Eq. (59) but it is a causal transform, that is, its inverse vanishes for negative times, thereby fulfilling the first condition of Titchmarsh's theorem. The Plemelj formulas in Eqs. (49a) and (49b), constituting the second and third conditions of Titchmarsh's theorem, are also satisfied by $Q(\omega)$ for all non-integer values of β. Thus, $Q(\omega)$ in Eq. (60) is an example of a function that fulfills all three conditions of Titchmarsh's theorem but without satisfying the fundamental requirement of square-integrability upon which the theorem is predicated.

VII. CONCLUDING REMARKS

For linear, plane-wave propagation in a homogeneous dispersive medium, the wavenumber, $-iK(\omega)$, is a causal transform, that is, the Fourier transform of the impulse response, $k(t)$, which is zero for $t < 0$. The imaginary part of the wavenumber is the attenuation, $\alpha(\omega)$, which, in certain materials encountered in underwater acoustics, medical ultrasound and geophysics is observed to take the form of a frequency power law, $|\omega|^{\beta}$, over an extended bandwidth. Although the exponent, β, may be any real number, in practice it is often found to lie between zero and two. Of course, beyond the measurement bandwidth, the behavior of the attenuation is unknown but one hypothesis is that the frequency power law is obeyed over an indefinitely wide frequency range extending from zero to infinity. The implications of the infinite-bandwidth hypothesis are explored in this paper.

A measure of the dispersion is provided by the real part of the wavenumber, $\omega/c(\omega)$, where $c(\omega)$ is the frequency-dependent phase speed. To satisfy the causality condition on $k(t)$, the Fourier components of the real and imaginary parts of $-iK(\omega)$ must combine in such a way as to ensure that $k(t)$ vanishes for $t<0$. The problem is to find an expression for the dispersion, the real part of $K(\omega)$, when the attenuation scales as $|\omega|^{\beta}$ over the frequency interval $(-\infty,\infty)$. The dispersion formula for the wavenumber $K(\omega)$ is derived in this paper, for β any real number not an integer, in three fundamentally different ways.

The first approach is based upon the fact that causality requires the impulse response function, $k(t)$, to vanish for $t<0$. By calling on the theory of generalized functions, a Fourier inversion is applied to $\alpha(\omega)$ to obtain an even function of time, from which an odd function of time is constructed in such a way as to ensure that $k(t)$ satisfies causality. A Fourier transform of the odd function of time then returns an expression for the dispersion, allowing the wavenumber to be specified to within the scaling constant, $\alpha_0(\beta)$ in Eq. (1).

Causality alone is not sufficient to establish the form of the scaling constant, $\alpha_0(\beta)$. Instead, in the second approach to the dispersion problem, a wave equation is introduced, based on the mechanism of strain hardening, the solution of which is the dispersion associated with an attenuation scaling as $|\omega|^{\beta}$ over an infinite bandwidth. In this solution, the scaling constant $\alpha_0(\beta)$ is fully determined [Eq. (30)], yielding the complete expression for the wavenumber, $K(\omega)$ [Eq. (31)].

In the third approach, the Kramers-Krönig dispersion relations with no subtractions, otherwise known as the Plemelj formulas [Eq. (49)], provide the basis of the solution for $K(\omega)$. In essence, the two Plemelj formulas express the real (imaginary) part of a causal transform as the Hilbert transform of the imaginary (real) part. They are usually established on the assumption that the causal transform in question is square-integrable and analytic, neither of which holds for an attenuation scaling as $|\omega|^{\beta}$. Before applying the Plemelj formulas to the function $|\omega|^{\beta}$, it is shown, without resorting to analytic continuation into the complex frequency plane, that they are in fact valid for an attenuation obeying a frequency power law, although the integrals converge only for β in the interval (0, 1). A continuation technique is introduced to extend the validity of the integrals to all non-integer values of β, including those lying outside the interval (0, 1). The eventual result is exactly the same expression [Eq. (58)] for the wavenumber, $K(\omega)$, as obtained in the first approach [Eq. (31)] based on the causal requirement that $k(t)$ must vanish for $t<0$. As with the causal argument, the Plemelj formulas do not return the scaling constant $\alpha_0(\beta)$, which is only available from the strain-hardening wave equation.

The dispersion formula for the wavenumber [Eq. (31) and Eq. (58)], as determined from all three approaches, is well behaved in the sense of showing no singularities for β any real number not an integer. Integer values of β give rise to non-physical behavior in the form of an infinite sound speed at finite frequencies, or an attenuation of zero at all frequencies, or a dispersion that is indeterminate. From a number of independent arguments, it is concluded that all integer values of β are non-realizable and hence are inadmissible. Physically, this means that, with β an integer, it is not possible for the Fourier components of the attenuation and dispersion to combine in such a way as to make the impulse response vanish for $t<0$.

There are also restrictions on the non-integer values that β may take. The dispersion formula [Eq. (31) and Eq. (58)] must satisfy two existence conditions: the phase speed and the attenuation must both be even functions of frequency; and the phase speed and the attenuation must both be positive. These two constraints determine the ranges of non-integer β that are physically realizable. It turns out that the admissible values of β depend on whether β is greater than or less than unity. For $\beta>1$, the admissible intervals are $(4n+4,\ 4n+5)$, and for $\beta<1$, they are $(-4n+0.5,\ -4n+1)$, where integer $n\geq 0$. Thus, the admissible values of β that are clustered around $\beta=0$ fall in the intervals $(-7.5,-7)$, $(-3.5,-3)$, (0.5, 1), (4, 5), and (8, 9). There is no upper or lower limit on the value that β may take.

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APPENDIX: FOURIER TRANSFORMS OF GENERALIZED FUNCTIONS

Table I lists the Fourier transforms of the generalized functions used in the text. It is based on Table I in Lighthill, although the notation is slightly different in order to be consistent with the conventions used above. Thus, the Fourier transform of a function of time, $y(t)$, is

$$Y(\omega)=\int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt \tag{A1}$$

and the inverse transform is

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega. \tag{A2}$$

Each entry in Table I is the transform of the product of the expression at the head of its column with the expression at the head of its row. For example, the Fourier transform of t^n

TABLE I. Fourier transforms of the generalized functions used in the text. [After Lighthill (Ref. 19), Table I, p. 43.]

	1	sgn(t)
$\lvert t\rvert^{\mu}$	$\{2\cos\pi(\mu+1)/2\}\mu!\lvert\omega\rvert^{-\mu-1}$	$\{-2i\sin\pi(\mu+1)/2\}\mu!\lvert\omega\rvert^{-\mu-1}\mathrm{sgn}(\omega)$
t^n	$(2\pi i)^n\delta^{(n)}(\omega)$	$2(n!)(i\omega)^{-n-1}$
t^{-m}	$-\pi i\dfrac{(-i\omega)^{m-1}}{(m-1)!}\mathrm{sgn}(\omega)$	$-2\dfrac{(-i\omega)^{m-1}}{(m-1)!}\{\ln\lvert\omega\rvert+C\}$
$t^n\ln\lvert t\rvert$	$-\pi i\dfrac{n!}{(i\omega)^{n+1}}\mathrm{sgn}(\omega)$	$-2\dfrac{n!}{(i\omega)^{n+1}}\{\ln\lvert\omega\rvert-\psi(n)\}$

sgn(t) is $2(n!)(i\omega)^{-n-1}$. In Table I, μ stands for any real number not an integer, n for any integer ≥ 0, m for any integer > 0, C is an arbitrary constant, and $\psi(x) = (d/dx)\ln x!$.

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