

An analytical solution for benchmark problem 1: The "ideal" wedge

Michael J. Buckingham
Mission Management Department, Royal Aerospace Establishment, Farnborough,
Hampshire GU14 6TD, England

Alexandra Tolstoy
Acoustics Division, Code 5120, Naval Research Laboratory, Washington DC 20375-5000

(Received 1 December 1988; accepted for publication 3 October 1989)

Results from an exact analytical solution for the field in a wedge with pressure-release boundaries (benchmark problem 1) are presented in the form of transmission loss as a function of horizontal range from the source. These computed curves contain no significant error, and thus may be used as a primary benchmark for establishing the accuracy of range-dependent, numerical propagation models.

PACS numbers: 43.30.Bp, 43.20.Mv

INTRODUCTION

Over the past decade or so, several computer codes have been developed for calculating the sound field in complex underwater environments. These algorithms are based on numerical solutions of the wave equation (or related equations) and are very complicated, generally requiring long run times to arrive at a final estimate of the field. Most, if not all, of these codes are based on some more or less subtle form of approximation. After many hours of CPU time, when the estimate for the field is finally produced, the question inevitably arises: How accurate is the result?

There is no simple answer to this enquiry because, for most range-dependent ocean channels, no reference solution exists. This problem has been recognized by the ocean acoustics community for some time, and was addressed recently in two "benchmark" sessions at consecutive meetings of the Acoustical Society of America. At the first of these meetings, three range-dependent problems were specified, benchmarks 1, 2, and 3, to be used as test cases for comparing the various propagation models, one with another. The details of these three problems are given by Jensen and Ferla¹ in this issue of the *Journal of the Acoustical Society of America*.

The purpose of this paper is to present an analytical solution for benchmark problem 1, which is the two-dimensional, "ideal" wedge problem: The acoustic field is required in a wedge, with pressure-release boundaries, which contains a line source parallel to the apex. This is one of the few range-dependent problems with an exact analytical solution. The form of this solution is outlined below, and transmission loss results evaluated from it are given graphically. Comments are included about the evaluation procedure and the care that was exercised to eliminate errors.

I. THE ANALYTICAL SOLUTION

Figure 1 shows the geometry of the wedge and the cylindrical coordinates used in the analysis. The medium supporting the field is a homogeneous fluid, the line source is parallel to the apex, and the boundaries are plane, pressure-release surfaces.

The procedure used to obtain an expression for the field is to apply two integral transforms to the inhomogeneous Helmholtz equation, a finite Fourier sine transform, and a Hankel transform. On taking the inverse transforms, the velocity potential is found to be the following sum of uncoupled normal modes:

$$\Psi = \frac{2}{\theta_0} \sum_{m=1}^{\infty} I_{\nu}(r, r') \sin(\nu\theta) \sin(\nu\theta'), \quad (1)$$

where θ_0 is the wedge angle, r and r' are the ranges of the receiver and source from the apex of the wedge, θ and θ' are the angular depths of the receiver and source measured about the apex, and

$$\nu = m\pi/\theta_0. \quad (2)$$

The mode coefficients in Eq. (1) are

$$I_{\nu}(r, r') = \int_0^{\infty} \frac{p}{p^2 - k^2} J_{\nu}(pr) J_{\nu}(pr') dp, \quad (3)$$

where k is the wavenumber of the source radiation and $J_{\nu}(\cdot)$ is a Bessel function of the first kind of order ν . The integral in Eq. (3) is Hankel's discontinuous integral, which is equal to the product of a Bessel function and a Hankel function.² When this result is substituted into Eq. (1), the final expression for the field is found to be

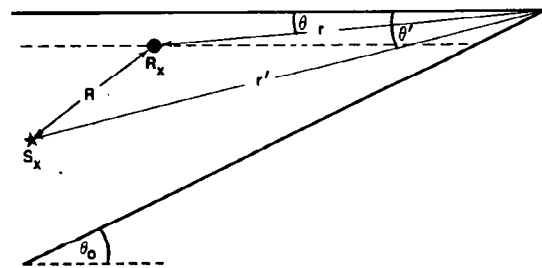


FIG. 1. Coordinates of the source, S_x , and receiver, R_x , in the ideal wedge. The wedge angle is θ_0 , and the radial distance between the source and receiver is R .

$$\Psi(r,r',\theta,\theta') = \frac{i\pi}{\theta_0} \sum_{m=1}^{\infty} J_{\nu}(kr_{<}) H_{\nu}^{(1)}(kr_{>}) \times \sin(\nu\theta)\sin(\nu\theta'), \quad (4)$$

where $r_{<} = \min(r,r')$, $r_{>} = \max(r,r')$, $i = (-1)^{1/2}$, and $H_{\nu}^{(1)}(\cdot)$ is the Hankel function of the first kind of order ν .

The expression for the velocity potential in Eq. (4) is exact and valid for all wedge angles (i.e., it is not restricted to wedge angles that are submultiples of π). Thus it includes the effect of diffraction by the apex of the wedge, as well as the image component of the field.

For comparison with other solutions for the field in the wedge, the expression in Eq. (4) is to be normalized to the field at 1 m from a line source in an unbounded homogeneous medium. Now, the velocity potential of the field generated by a line source in an infinite medium is

$$\Phi(R) = (i/4)H_0^{(1)}(kR), \quad (5)$$

where R is the radial distance from the source to the field point and $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind of order zero. This result, which depends on just one spatial coordinate R , is well known.³ One method of deriving it is to apply a Hankel transform to the Helmholtz equation, followed by the inverse transformation. Equation (5) represents cylindrical spreading and shows a logarithmic singularity at the origin, $R = 0$.

With R set equal to 1 m, the normalized field in the wedge is

$$\Lambda = |\Psi(r,r',\theta,\theta')|/|\Phi(1)|, \quad (6)$$

and the transmission loss in dB is

$$TL = 20 \log_{10}(\Lambda). \quad (7)$$

This expression was evaluated for a fixed receiver depth as a function of horizontal range from the source to the apex.

II. THE COMPUTATIONS

Full details of the parameter values for benchmark problem 1 are listed in Table I of Ref. 1, and the source/receiver configuration in the wedge is illustrated in Fig. 1 of Ref. 1. Note that the sound speed of interest is $c = 1500$ m/s, the frequency is 25 Hz, and the wedge angle is $\theta_0 = \arctan(1/20) = 2.862405^\circ$. The source is at a horizontal range of 4 km from the apex of the wedge, at a depth of 100 m. This is midway down the water column, which is 200 m deep at the source position.

Equation (7) was evaluated with these parameter values on a VAX 11/750 computer, using double precision throughout, for 1000 evenly spaced horizontal range points at a depth of 30 m. A standard Bessel function package known as VAXMATH⁴ (also known as SLATEC) and asymptotics were used to evaluate the Hankel and Bessel functions. With 100 modes in the summation for the field in the wedge, the calculation of TL took less than 2 min. of CPU time.

Computational difficulties can arise with the imaginary part of the Hankel function [i.e., the Neumann function $Y_{\nu}(x)$] when the order ν becomes large relative to the argument x : for a fixed value of x , $Y_{\nu}(x) \rightarrow -\infty$ as $\nu \rightarrow \infty$. We tested the VAXMATH calculations against tabulated results published by Abramowitz and Stegun⁵ for a wide range

of values, including $(x,\nu) = (1,20), (2,30), (5,50), (10,50), (40,50), (50,50), (50,100),$ and $(100,100)$, and found exact agreement. We also tested the Debye asymptotic expansions for large ν (Eqs. 9.3.7 and 9.3.8 in Abramowitz and Stegun⁵), using only the first two terms, and found very good agreement (to three significant figures) with the corresponding tabulated values in Ref. 5. The latter checks were performed because, when the value of Y_{ν} dropped below -10^{10} , we switched from VAXMATH to Debye asymptotics.

In the benchmark problem, only odd-order modes are excited, since the source is symmetrically placed in the channel at middepth. Moreover, since $kr' \approx 400$ and $\nu = (62.9 \times \text{mode number})$, we see that only modes 1, 3, and 5 will propagate through the wedge. However, in the immediate vicinity of the source, many more terms must be included in the mode sum to achieve acceptable accuracy, that is, to give the correct nearfield behavior. We considered as many as 1000 modes for field computations at the source depth, in an attempt to examine nearfield and normalization effects, but there was no perceptible difference between transmission loss (TL) curves, plotted on a scale of 4 dB/cm, based on sums containing 100 modes and 1000 modes. Moreover, for $R > 120$ m (i.e., two wavelengths at 25 Hz), where R is the range from the source, these plots were no different from a plot derived from a summation of only five modes. Thus we feel confident that our results based on sums of 100 modes contain no significant truncation errors.

III. RESULTS

Figure 2 shows the transmission loss calculated from Eq. (7) for benchmark problem 1. TL is shown as a function of horizontal range measured from the source of increments of 4 m, at a depth of 30 m. The field shows rapid spatial fluctuations at ranges up to 3.4 km, which is the cutoff range (measured from the source) of the first mode. The structure of these fluctuations changes at a range of 2.2 km, which is the cutoff range of the third mode. Thus at ranges between 2.2 and 3.4 km, only the first mode contributes to the field, which shows a more regular structure than at shorter ranges, where two or more modes interfere.

In the region where only the first mode is present, the

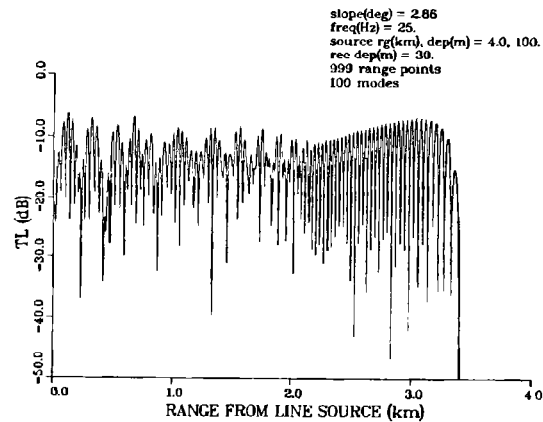


FIG. 2. Transmission loss as a function of horizontal range from the source, calculated from Eq. (7) for a receiver depth of 30 m a frequency of 25 Hz, and a wedge angle of 2.86 deg.

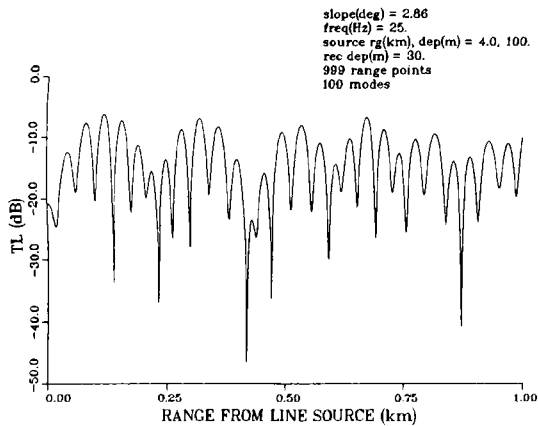


FIG. 3. Expanded view of the transmission loss in Fig. 2, over the range 0–1 km.

peaks and troughs in the field are due to *intramode interference*. This phenomenon is characteristic of range-dependent channels.^{6,7} It arises because a mode consists of two field components, one traveling up slope and the other down slope, which interfere to give the observed effect. At ranges greater than 3.4 km (i.e., close to the apex), there is essentially no energy in the field because the water depth at such ranges is not sufficient to support even the lowest-order mode. (As it happens, the intercept of the 30-m depth line with the bottom boundary of the wedge coincides with the cutoff range of the first mode. Thus the cutoff region is not depicted in Fig. 2, since at a range greater than 3.4 km, the receiver is outside the wedge domain.)

Figure 3 shows an expanded view of the field in Fig. 2, spanning the range out to 1 km and calculated using 1000 range points. This is useful for a detailed comparison with the results obtained from other models.

IV. CONCLUDING DISCUSSION

The solution for the field in the ideal wedge shown in Figs. 2 and 3 has been carefully checked and is exact: The peaks and troughs fall in the correct positions, the relative heights of the peaks are also correct, as are the absolute levels of the curve. Thus these transmission loss curves may be regarded as a *primary* range-dependent benchmark. They may be used for comparison with the predictions of range-dependent numerical codes, to establish the accuracy of the latter.

Jensen and Ferla¹ have examined three numerical models in connection with all three benchmark problems. Two of these models are based on the parabolic equation,⁸ and are thus unsuitable for estimating the field in the ideal wedge (because backscattering is ignored in the parabolic approximation). The third model is a complete two-way solution of

the elliptic wave equation known as “COUPLE”.⁹ It is formulated in terms of stepwise-coupled normal modes and is valid for a range-dependent, fluid medium. COUPLE is CPU-time intensive, as illustrated by the comparative figures in Table III of Ref. 1.

On comparing the results of COUPLE shown in Figs. 3(b) and 4 of Ref. 1 with our calculations of the field in the ideal wedge, we see that there is excellent overall agreement. The positions of the peaks and nulls are the same, with just a slight occasional discrepancy between the height of interference peaks of less than 1 dB. Moreover, the absolute levels of the curves are in good agreement, to within a fraction of a dB. Since the ideal wedge problem is a particularly stringent test of any numerical propagation code, this agreement with the exact analytical solution establishes COUPLE as a *secondary* range-dependent benchmark. Thus COUPLE may be used for comparison with computationally efficient numerical codes, such as those based on the parabolic equation, to assess their accuracy in the context of realistic, range-dependent ocean channels, where backscattering effects are negligible.

In conclusion, although the analytical solution of the ideal wedge problem is not very realistic in terms of the complex ocean environment, it is extremely useful as a primary, range-dependent benchmark. Since it is exact, it may be used to establish the accuracy of a computationally demanding numerical code, such as COUPLE. The latter may then be used as a secondary benchmark against which other, less computationally intensive, algorithms can be tried and tested in the context of realistic ocean channels.

¹ F. B. Jensen and C. M. Ferla, “Numerical solutions of range-dependent benchmark problems in ocean acoustics,” *J. Acoust. Soc. Am.* **87**, 1499–1510 (1990).

² G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge U. P., Cambridge, MA, 1958), p. 429.

³ P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Eq. 11.2.23.

⁴ W. R. Boland, J. Chow, J. Ford, and A. B. White, Jr., “VAXMATH: A Mathematical Software Library for the VAX,” *Proceedings of the Digital Computer Users Society*, Fall, 1982.

⁵ M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (National Bureau of Standards, Washington, DC, 1972), Appl. Math. Ser. No. 55.

⁶ M. J. Buckingham, “Acoustic propagation in a wedge-shaped ocean with perfectly reflecting boundaries,” in *Hybrid Formulation of Wave Propagation and Scattering*, edited by L. B. Feisen (Nijhoff, Dordrecht, The Netherlands, 1984), pp. 77–105; and NRL Rep. 8793 (1984).

⁷ M. J. Buckingham, “Theory of acoustic propagation around a conical seamount,” *J. Acoust. Soc. Am.* **80**, 265–277 (1986).

⁸ F. D. Tappert, “The parabolic equation method,” in *Wave Propagation and Underwater Acoustics*, edited by J. B. Keller and J. S. Papadakis (Springer, New York, 1977).

⁹ R. B. Evans, “A coupled mode solution for acoustic propagation in a waveguide with stepwise depth variations of a penetrable bottom,” *J. Acoust. Soc. Am.* **74**, 188–195 (1983).