Spherically symmetrical acoustic propagation across a fluid/fluid boundary

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The problem of determining the spherically symmetrical field generated by an acoustic point source at the center of a fluid sphere that itself is immersed in a second fluid medium is discussed. A method of solving for the field in the internal and external fluid regions is introduced, based on finite Hankel transforms. These transforms and their inverses are constructed in such a way that the boundary conditions (continuity of pressure and normal component of particle velocity) at the spherical interface are satisfied. In particular, the inverse transforms are integrals; that is, there is a continuum of radial wavenumbers. This contrasts with the more usual formalism, involving a Fourier–Bessel series taken over a discrete distribution of radial wavenumbers. The latter type of inversion formula is suitable only for problems involving relatively simple boundary conditions (e.g., Dirichlet or Neumann). Using the new technique, an exact analytical solution is given for the acoustic field in both the sphere and the surrounding fluid.

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INTRODUCTION

In a recent theoretical investigation[1] of ambient noise in the Arctic Ocean below the marginal ice zone, the principal source of the noise was assumed to be the clashing together of ice floes on the sea surface. As a starting point in the analysis of the noise, it was necessary to determine the field generated by a single ice floe when it experienced a collision with another floe. Each ice floe was modeled as a fluid sphere, with a (fictitious) point source of sound located at its center. This is undoubtedly an oversimplification as far as the ambient noise modeling is concerned, but it led to the interesting problem of deriving a solution for the field generated by the source, both inside and outside the sphere, on the assumption that the sphere is totally immersed in another fluid (the ocean) with different acoustic properties.

The impedance mismatch at the boundary between the fluid sphere and the ocean introduces a degree of complexity to the problem that makes pursuing it worthwhile: Partial reflection and partial transmission occur at the spherical interface. (The solution for the field in a sphere with a perfectly reflecting boundary is well known.[2]) A method of solution is presented here that itself has intrinsic interest, since it is based on the application of finite Hankel transforms. Such transforms have not been widely applied to problems in acoustics, unlike conventional Hankel transforms taken over an infinite range, which are well known and have been applied with advantage recently to several three-dimensional acoustic propagation problems.[3–5]

Sneddon[6,7] introduced some simple types of finite Hankel transforms, which are applicable to problems with Dirichlet, Neumann, or mixed (i.e., impedance) boundary conditions. The inversion formulas of the transforms discussed by Sneddon are Fourier–Bessel series. The situation is analogous to finite Fourier transforms whose inversion formulas are Fourier series. In our problem with the fluid-loaded fluid sphere, the boundary conditions are that the pressure and normal component of velocity should be continuous across the spherical interface between the two fluids. These conditions are not of the impedance type. In this case the appropriate inversion formulas for the finite Hankel transforms are integrals rather than series. These inversion integrals and the differential properties of the finite Hankel transforms are discussed in the Appendix.

I. ANALYSIS OF THE FIELD IN THE TWO FLUIDS

Imagine a fluid sphere of radius a (region 1) with density ρ_1 and sound speed c_1 immersed in an infinite fluid (region 2) with density ρ_2 and sound speed c_2, as shown in Fig. 1. A point, impulsive source is located at the center of the sphere with source strength $S\delta(t)$, where $\delta(\ )$ is the Dirac delta function and t is time. We wish to solve the wave equation for the velocity potential of the field in regions 1 and 2, subject to the appropriate boundary conditions at the surface of the sphere.

Suppose that $g_1(t,r)$ and $g_2(t,r)$ are the velocity potentials in regions 1 and 2, respectively, with Fourier transforms $G_1(j\omega,r)$ and $G_2(j\omega,r)$, where r is range out from the source, ω is angular frequency, and $j=(-1)^{1/2}$. On Fourier transforming the wave equation (with respect to time), we obtain the Helmholtz equation for regions 1 and 2. On expressing the Helmholtz equation in spherical polar coordinates and making the familiar substitution

$$G = w/r^{1/2}, \tag{1}$$

we obtain Bessel's equation involving the function $w \equiv w(r)$, which is now the function we wish to determine. Note that the problem is one dimensional since, through symmetry, the field shows no angular variation. (By moving the source away from the center of the sphere, we could, of course,

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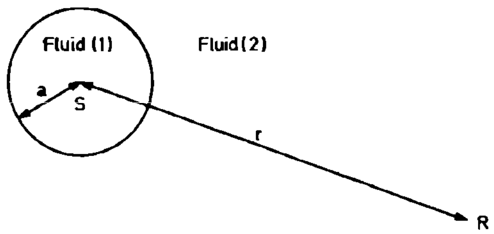

FIG. 1. Spherical domain of radius a containing a fluid (1) with density ρ_1 and sound speed c_1, immersed in a second, infinite fluid (2) with density ρ_2 and sound speed c_2. An acoustic point source S is at the center of the sphere, and a receiver R is at a range r from the source. Although R is shown in fluid (2), the analysis gives the field in both domains.

introduce an angular dependence, and the final solution for the field would then involve a summation of Legendre functions. Since we are primarily interested in the range dependence, in connection with the finite Hankel transforms, we consider here only the symmetrical case. Thus our solution will consist of just the zeroth-order term rather than an infinite sum of spherical harmonics.)

Bessel's equation for the two regions can now be written as follows:

$$\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} - \frac{w_1}{4r^2} + k_1^2 w_1 = -S\frac{\delta(r-r')}{4\pi r'^{3/2}}, \quad r \leqslant a \tag{2}$$

and

$$\frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r}\frac{\partial w_2}{\partial r} - \frac{w_2}{4r^2} + k_2^2 w_2 = 0, \quad r > a, \tag{3}$$

where the source range r' on the right of Eq. (2) will eventually be set equal to zero. In Eqs. (2) and (3), subscripts 1 and 2 still identify quantities associated with the internal and external regions, respectively. The boundary conditions at the surface of the sphere are

$$\rho_1 w_1(a) = \rho_2 w_2(a), \tag{4}$$

and

$$\left.\frac{\partial w_1}{\partial r}\right|_{r=a} - \left.\frac{\partial w_2}{\partial r}\right|_{r=a} = \frac{w_1(a) - w_2(a)}{2a}, \tag{5}$$

representing continuity of pressure and normal component of velocity, respectively.

Taking the inner region $0 \leqslant r \leqslant a$ first, we find from Eq. (2) and the results in the Appendix that the finite Hankel transform of w_1 is

$$w_{1_p} = [F_1(a,p) - (S/4\pi)(2/\pi)^{1/2}p^{1/2}]/(k_1^2 - p^2), \tag{6}$$

where, in the source term, we have taken the limit as $r' \to 0$, and

$$F_1(a,p) = a w_1(a)[J_{1/2}'(pr)]_{r=a} - a[w_1'(r)]_{r=a} J_{1/2}(pa). \tag{7}$$

The primes in this expression denote differentiation with respect to r. On taking the inverse Hankel transform of Eq. (6) and replacing the half-integer order Bessel functions by trigonometric functions, we eventually find that

$$w_1(r) = \frac{1}{\pi r^{1/2}}\left(A_1\int_0^\infty \frac{p[\sin p(a+r) - \sin p(a-r)]}{(k_1^2 - p^2)}dp - B_1\int_0^\infty \frac{\cos p(a-r) - \cos p(a+r)}{(k_1^2-p^2)}dp - \frac{S}{2\pi}\int_0^\infty \frac{p\sin pr}{(k_1^2-p^2)}dp\right), \tag{8}$$

where the constants A_1 and B_1 are as follows:

$$A_1 = a^{1/2} w_1(a) \tag{9}$$

and

$$B_1 = a^{1/2}[w_1(a)/2a + w_1'(r)_{r=a}]. \tag{10}$$

The integrals in Eq. (8) are known forms[8]:

$$\int_0^\infty \frac{x\sin\alpha x}{(\beta^2 - x^2)}dx = -\frac{\pi}{2}\exp(-j\alpha\beta) \tag{11}$$

and

$$\int_0^\infty \frac{\cos\alpha x}{(\beta^2 - x^2)}dx = -\frac{\pi}{2j\beta}\exp(-j\alpha\beta). \tag{12}$$

When these results are used to evaluate Eq. (8), we find that

$$w_1(r) = (1/2r^{1/2})[(S/2\pi)\exp(-jk_1 r) + (2j/k_1)(k_1 A_1 - jB_1)\exp(-jk_1 a)\sin k_1 r]. \tag{13}$$

Setting $r = a$ in this result yields an expression for $w_1(a)$ in terms of the derivative of w_1 at $r = a$:

$$w_1(a) = [\exp(-jk_1 a)][Sk_1 a^{1/2}/2\pi + 2aw_1'(r)_{r=a}\sin k_1 a]/[2k_1 a - (1 + 2jk_1 a)\exp(-jk_1 a)\sin k_1 a]. \tag{14}$$

The next step is to go through a similar procedure for the external region where $r > a$. From Eq. (3), after Hankel transforming, taking the inverse and performing the integrations, we obtain

$$w_2(r) = (k_2 r^{1/2})^{-1}\exp(-jk_2 r) \times [A_2 k_2\cos k_2 a - B_2\sin k_2 a], \tag{15}$$

where the constants A_2 and B_2 are as follows:

$$A_2 = a^{1/2} w_2(a) \tag{16}$$

and

$$B_2 = a^{1/2}[w_2(a)/2a + w_2'(r)_{r=a}]. \tag{17}$$

Setting $r = a$ in Eq. (15) yields an expression for $w_2(a)$ in terms of the derivative of w_2 at $r = a$:

$$w_2(a) = \frac{-2a\exp(-jk_2 a)w_2'(r)_{r=a}\sin k_2 a}{[2ak_2 + \exp(-jk_2 a)(\sin k_2 a - 2ak_2\cos k_2 a)]}. \tag{18}$$

Equations (14) and (18) in conjunction with the boundary conditions in Eqs. (4) and (5) form a system of four simultaneous equations in the four unknowns $w_1(a)$, $w_2(a)$, $w_1'(r)_{r=a}$, and $w_2'(r)_{r=a}$. After some straightforward but rather tedious algebra, these four unknowns can be determined, allowing the functions $w_1(r)$ and $w_2(r)$ to be specified. The Green's functions $G_1(j\omega,r)$ and $G_2(j\omega,r)$ for the internal and external regions, respectively, are then found to be as follows:

$$G_1(j\omega,r) = (S/4\pi r)\{\exp(-jk_1 r) + jD^{-1}\exp(-jk_1 a)\sin(k_1 r) \times [k_2^{-1}\exp(-jk_2 a)\sin(k_2 a) + jk_1^{-1}N]\}, \quad (19)$$

where $r \leqslant a$, and

$$G_2(j\omega,r) = (S/4\pi r)(k_2 D)^{-1} b \exp(-jk_2 r)\sin(k_2 a), \quad (20)$$

where $r > a$. In these expressions, the parameter

$$b = \rho_1/\rho_2 \quad (21)$$

is the density ratio between regions 1 and 2, and the two functions N and D are as follows:

$$N = \exp(-jk_2 a)\sin(k_2 a)[b + (b-1)/(k_2 a)] \quad (22)$$

and

$$D = \exp(-jk_2 a)\sin(k_2 a) \times \{k_1^{-1}\sin(k_1 a)[jb + (b-1)/(k_2 a)] + k_2^{-1}\cos(k_1 a)\}. \quad (23)$$

Equations (19) and (20) are the results we have been seeking. They are quite general in that they give the velocity potential of the field in the two fluid regions for arbitrary density and sound speed ratios. As a check, it is easy to show that, in the special case when there is no acoustic mismatch at the boundary of the fluid sphere, Eqs. (19) and (20) reduce to the familiar form for the field due to a point source radiating into an infinite fluid medium.

II. NEUTRAL BUOYANCY

When the sphere is neutrally buoyant, as would approximately be the case for ice immersed in water, the expressions given above for the field simplify considerably. This is illustrated below for the field in the external region (the ocean), although similar arguments apply for the field within the sphere itself.

For the case of neutral buoyancy, the density ratio b is unity and the Green's function in Eq. (20) reduces to the form

$$G_2(j\omega,r) = (S/4\pi r)\exp[-jk_2(r-a)]\Omega(j\omega), \quad (24)$$

where

$$\Omega(j\omega) = [\cos(k_1 a) + j\gamma\sin(k_1 a)]^{-1}. \quad (25)$$

In this expression, γ is the ratio of the sound speeds in the two media:

$$\gamma = c_1/c_2 = k_2/k_1. \quad (26)$$

The spectrum of the Green's function in Eq. (24) is simply

$$|G_2(j\omega,r)|^2 = (S/4\pi r)^2|\Omega(j\omega)|^2, \quad (27)$$

where, from Eq. (25),

$$|\Omega(j\omega)|^2 = [\cos^2(k_1 a) + \gamma^2\sin^2(k_1 a)]^{-1}. \quad (28)$$

When γ is unity, there is no acoustic mismatch at the boundary of the sphere, the expression on the right of Eq. (28) is unity, and the field in Eq. (27) is independent of frequency, corresponding to simple spherical spreading.

In general, γ is not equal to unity and the field in Eq. (27) shows some frequency dependence through the function $\Omega(j\omega)$. The form of this frequency dependence is shown in Fig. 2, where $|\Omega(j\omega)|^2$ is plotted as a function of the dimensionless frequency $k_1 a$, for $\gamma = 2$. Note that the peaks in the spectrum occur when $k_1 a$ is a multiple of π and the troughs fall at odd multiples of $\pi/2$. This is true for any value of $\gamma > 1$, with the spectral periodicity continuing up to indefinitely high frequencies (because no losses are included in the analysis). It is readily shown from Eq. (28) that, when $\gamma < 1$, a similarly shaped spectrum is obtained, except that the peaks occur when $k_1 a$ is an odd multiple of $\pi/2$ and the troughs fall at multiples of π.

Although the spectral form in Fig. 2 has a certain curiosity value, it does not provide an immediate insight into the physical processes underlying the acoustic propagation. More can be gained from the time dependence of the pulse observed at the receiver.

III. THE PRESSURE PULSE AT THE RECEIVER

Equations (24) and (25) give the Fourier transform (with respect to time) of the velocity potential due to an impulsive source at the center of the neutrally buoyant sphere. The inverse Fourier transform of $G_2(j\omega,r)$ gives the velocity potential $g_2(t,r)$ of the pulse at the receiver:

$$g_2(t,r) = (2\pi)^{-1}\int_{-\infty}^{\infty} G_2(j\omega,r)\exp(j\omega t)d\omega. \quad (29)$$

To evaluate this inversion integral, it is convenient to express Eqs. (24) and (25) in the alternative form

$$G_2(j\omega,r) = [ST/(4\pi r)] \times [1 - V\exp(-2jk_1 a)]^{-1}\exp(-jk_2\chi), \quad (30)$$

where

$$\chi = (r-a) + (a/\gamma), \quad (31)$$

$$T = 2/(\gamma+1), \quad (32a)$$

and

$$V = (\gamma-1)/(\gamma+1). \quad (32b)$$

The parameters T and V have a simple interpretation: They are the plane-wave transmission and reflection coefficients, respectively, for normal incidence of an internal ray at the

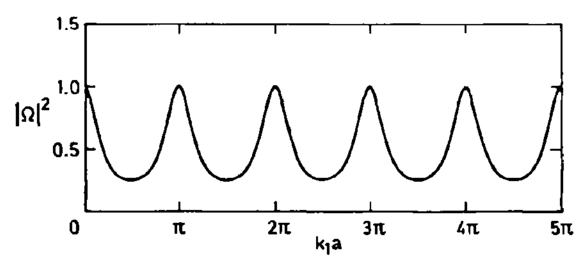

FIG. 2. The spectral function $|\Omega|^2$ in Eq. (28) plotted against $k_1 a$, with $\gamma = 2$.

spherical interface between the two fluid media. Now, the term containing the reflection coefficient in Eq. (30) can be expressed as a geometric series:

$$\frac{1}{1 - V\exp(-2jk_1a)} = \sum_{n=0}^{\infty} V^n \exp(-2jnk_1a). \quad (33)$$

When this is substituted back into Eq. (30), which, in turn, is substituted into Eq. (29), the integral for the velocity potential becomes

$$g_2(t,r) = \frac{ST}{(8\pi^2 r)} \sum_{n=0}^{\infty} V^n \times \int_{-\infty}^{\infty} \exp\left[j\omega\left(t - \frac{\chi}{c_2} - \frac{2na}{c_1}\right)\right] d\omega. \quad (34)$$

The integral in this expression is a well-known form of the Dirac delta function:

$$2\pi\delta(u) = \int_{-\infty}^{\infty} \exp jux\, dx, \quad (35)$$

and thus the final result for the velocity potential takes the form of an infinite sum of delta functions:

$$g_2(t,r) = \frac{ST}{(4\pi r)} \sum_{n=0}^{\infty} V^n \times \delta\left(t - \frac{(r-a)}{c_2} - \frac{(2n+1)a}{c_1}\right). \quad (36)$$

Note that, when $\gamma = 1$, that is when $c_1 = c_2 = c$, the reflection coefficient V is zero and the velocity potential reduces to the form

$$g_2(t,r) = [S/(4\pi r)]\delta[t - (r/c)]. \quad (37)$$

This is just the familiar retarded potential due to an impulsive point source in an infinite, homogeneous medium.

The pressure pulse associated with the velocity potential in Eq. (36) is

$$p_2(t,r) = \rho_2 \frac{\partial g_2}{\partial t} = \frac{\rho_2 ST}{(4\pi r)} \sum_{n=0}^{\infty} V^n \delta'\left(t - \frac{(r-a)}{c_2} - \frac{(2n+1)a}{c_1}\right), \quad (38)$$

where the prime denotes differentiation with respect to time. It is clear from Eq. (38) that the pressure pulse at the receiver consists of a succession of equispaced pressure doublets with decaying amplitudes, as shown schematically in Fig. 3. The first of these doublets is the direct arrival from the source to the receiver, which appears after a time $[(r-a)/c_2 + a/c_1]$. This is just the time taken to travel across the radius of the sphere from the center and then through the external medium to the receiver. Successive arrivals are separated by time intervals equal to $2a/c_1$, which is the time taken to cross the diameter of the sphere. These later arrivals are due to internal reflections at the spherical interface between the internal and external media. The decay in amplitude of these arrivals corresponds exactly to the attenuation introduced at each internal reflection.

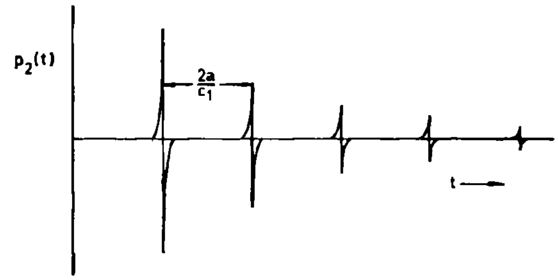

FIG. 3. Sketch showing sequence of equispaced pressure doublets in Eq. (38). The first pulse in the sequence is the direct arrival and the remainder arises from internal reflections at the spherical interface between the two fluid media. The time between successive arrivals is $2a/c_1$, which is just the travel time across the diameter of the sphere.

IV. CONCLUDING REMARKS

The problem of a point source at the center of a sphere, discussed above, is very straightforward since the symmetry ensures that the field depends on only one spatial variable (i.e., range out from the center). Integral transform techniques are hardly necessary to obtain a solution. The governing equations (2) and (3) are simply Bessel's equation, and the solutions for the functions w_1 and w_2 are obviously linear combinations of Bessel and Neumann functions of order one-half. Thus the corresponding functions G_1 and G_2 are spherical Bessel functions. Such functions are easily expressed in terms of trigonometric functions, which accounts for the trigonometric forms appearing in Eqs. (19) and (20). Four unknown constants are present in the linear combinations of Bessel and Neumann functions for w_1 and w_2, which may be determined from the two boundary conditions [Eqs. (4) and (5)], a radiation condition, and the requirement that the field show the correct form of singularity at the source point. This conventional approach to the problem leads to the same result for the field as that in Eqs. (19) and (20), obtained by the integral transform method, with a similar amount of algebra being required. Thus, in this case, there is no particular advantage to be gained from using the integral transform technique.

More generally, however, with more complex geometries, the field is multidimensional. For example, the field would be two dimensional in the sphere problem if the source were located away from the center. In multidimensional problems, the governing equations involve two or three spatial variables, and to write solutions for the field simply by inspection may require intuitive powers verging on the supernatural. In contrast, a sequence of two or three integral transforms applied to the governing equations followed by the appropriate inverses, provides a mechanistic technique for solving for the field in which the boundary conditions are automatically taken into account.

The power of the sequential transform approach in solving partial differential equations has been demonstrated recently in connection with a time harmonic point source in a wedge shaped domain. This classic, three-dimensional problem has been of interest since before the beginning of the century. It was finally solved exactly using a sequence of three integral transforms.[3]

In radiation problems showing spherical or cylindrical symmetry, it is fairly common to encounter Hankel transforms taken over an infinite range. When a boundary occurs along the range coordinate, separating two regions with different acoustic (or electromagnetic) properties, a conventional Hankel transform is no longer appropriate and, instead, finite Hankel transforms should be employed. The purpose of this article is to introduce a general form of finite Hankel transform, and to illustrate its properties in the context of a relatively simple, one-dimensional problem showing spherical symmetry. It should be noted that, in general, the inverse of the finite Hankel transform is an integral form rather than a Fourier–Bessel series. The integral form of the inverse transform allows moderately complicated boundary conditions to be handled (for example, continuity of pressure and normal component of velocity), whereas the series form is appropriate only for the special case of simple impedance-type boundaries.

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APPENDIX: PROPERTIES OF FINITE HANKEL TRANSFORMS

The finite Hankel transform of the function $w(r)$ is defined over the interval $\alpha \leqslant r \leqslant \beta$, with $\alpha \geqslant 0$, as follows:

$$w_p = \int_\alpha^\beta r w(r) J_\mu(pr) dr, \tag{A1}$$

where $J_\mu(\ )$ is the Bessel function of the first kind of order μ. The inverse transform is

$$w(r) = \int_0^\infty p w_p J_\mu(pr) dp, \quad \text{for } \alpha \leqslant r \leqslant \beta,$$
$$= 0, \quad \text{otherwise.} \tag{A2}$$

This result is proved with the aid of the Bessel function closure relation:

$$\int_0^\infty p J_\mu(pr) J_\mu(pr') dp = \frac{\delta(r-r')}{r}. \tag{A3}$$

The procedure is to multiply both sides of Eq. (A1) by $pJ_\mu(pr')$, integrate over p from zero to infinity, and use the sampling property of the delta function to arrive at the result in Eq. (A2).

Note that the inversion formula in Eq. (A2) involves an integral, rather than one of the various series discussed by Sneddon.[7] A series form for the inversion is appropriate when the boundary conditions are either Neumann or Dirichlet (or mixed). However, the interface between two fluids, such as that considered in the main text of this article, is not an impedance boundary, and the inversion integral in Eq. (A2) is the appropriate form to use.

The differential operator

$$\Delta_\mu \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu^2}{r^2} \tag{A4}$$

appearing in the Helmholtz equation shows the following finite Hankel transform:

$$\int_\alpha^\beta r \Delta_\mu w J_\mu(pr) dr$$
$$= \beta w'(\beta) J_\mu(p\beta) - \alpha w'(\alpha) J_\mu(p\alpha)$$
$$- \beta w(\beta) J_\mu{}'(p\beta) + \alpha w(\alpha) J_\mu{}'(p\alpha) - p^2 w_p, \tag{A5}$$

where the primes denote differentiation with respect to r. The result in Eq. (A5) is obtained by the familiar technique of integrating by parts twice and making use of the equality between the terms in Bessel's equation. When α and β are zero and infinity, respectively, the integrated parts vanish and Eq. (A5) reduces to the usual form for the Hankel transform of $\Delta_\mu w$.

The integrated parts in Eq. (A5) include explicitly the boundary conditions at the upper and lower ends of the integration interval. As the analysis of the sphere, discussed in the main text, adequately demonstrates, this formulation provides a mechanistic way of handling problems with moderately difficult boundary conditions.

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