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On the response of steered vertical line arrays to anisotropic noise

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An approximate method is presented for evaluating, through the noise gain function, the response of a steered vertical line array of acoustic sensors to anisotropic, plane-wave noise fields. On the basis of the high-N approximation a closed form solution is obtained for the noise gain function, even for the general case of arbitrary anisotropy. The main features on the noise gain curves are discussed and interpreted in terms of conventional beam-forming concepts.

1. INTRODUCTION

In a well-designed underwater detection system in which self-noise is minimal, the limit of performance is often set by the ambient noise field in the ocean. This limit will depend, in general, on the angular response of the detector and the anisotropy of the noise. The purpose of this paper is to consider the response of an unshaded vertical line array (v.l.a.) of equidistant sensors to noise fields showing arbitrary anisotropy.

The noise is treated on the assumption that it is spatially homogeneous over the aperture of the array, and can be represented as a linear superposition of plane waves propagating in all directions. The response of the v.l.a. to the ambient noise is expressed in terms of the noise gain of the array, which is defined here in such a way that its *product* with the signal gain gives the array gain. The noise gain is normalized so that, when the noise fluctuations at different hydrophones in the array are completely incoherent, its value is unity irrespective of the number, N, of sensors in the system; in this case the array gain is simply equal to the signal gain of the v.l.a. When the noise gain is greater than unity, corresponding to the rejection of noise by the array, the array gain is enhanced; and conversely, when the spatial coherence of the noise is such that the noise gain is less than unity, the array gain is reduced.

The normalization of the noise gain is important in that it allows the high-N approximation to be introduced, in which the array of N equidistant elements is replaced, for the purpose of calculating the noise gain, by a notional array with similar inter-element spacing but having an infinite number of elements. For a v.l.a. with five or more elements this is a good approximation, having the advantage of providing a closed-form solution for the noise gain function which is valid for arbitrary anisotropy and from which some insight into the physics of the array response to the noise may be derived.

2. The noise gain function

The noise gain of a v.l.a. of N equidistant sensors is defined as follows:

$$G_N(\omega, \theta') = \left[\frac{1}{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \{H_p(\mathrm{j}\omega)\, H_q^*(\mathrm{j}\omega)\, \Gamma_{pq}(\omega)\}\right]^{-1}, \tag{1}$$

where the asterisk denotes complex conjugate, ω is the angular frequency, θ' is the angle between the centre of the primary beam and the zenith, Γ_{pq} is the normalized cross-spectral density between the noise fluctuations at the pth and qth hydrophones in the array, and the functions H_p, H_q represent (in the frequency domain) the beam steering delays applied to the pth and qth hydrophones. Assuming that these delays are arranged in arithmetic progression, the product of the steering functions in (1) can be written in the form

$$\exp\{\mathrm{j}(p-q)\,(\omega l/c)\cos\theta'\},$$

where l is the inter-element separation and c is the speed of sound in the medium. This expression depends only on the separation of the pth and qth elements and not their actual spatial positions. In the same way, in a spatially homogeneous noise field, Γ_{pq} also depends only on the separation of the pth and qth elements in the array. Thus, bearing in mind that the normalized cross-spectral density is Hermitian, the double summation in (1) can be replaced with a single summation over the index $\kappa = (p-q)$ as follows:

$$G_N(\overline{\omega}, \theta') = \left[1 + 2\,\mathrm{Re} \sum_{\kappa=1}^{N} \left\{\left(\frac{N-\kappa}{N}\right) \Gamma_\kappa \exp\{\mathrm{j}k\overline{\omega}\cos\theta'\}\right\}\right]^{-1}, \tag{2}$$

where $\Gamma_\kappa \equiv \Gamma_{pq}$ and $\overline{\omega} = \omega l/c$ is the normalized angular frequency. The change of variable from ω to $\overline{\omega}$ in (2) should be indicated by some distinguishing mark on the function G_N but, since no confusion is likely to arise, such a mark has been omitted.

The noise gain of the array for any steering condition can be evaluated from (2) provided Γ_κ, representing the anisotropy of the noise, is known. In general the noise gain depends on the number of sensors in the array; and hence, for a given noise field and fixed steering delays, equation (2) gives rise to a family of curves, each member of which is associated with a particular value of N. For $N \geqslant 5$ all the curves in this family closely approximate the form

$$G_\infty(\overline{\omega}, \theta') = \left[1 + 2\,\mathrm{Re} \sum_{\kappa=1}^{\infty} \left\{\Gamma_\kappa \exp(\mathrm{j}k\overline{\omega}\cos\theta')\right\}\right]^{-1}, \tag{3}$$

which is the limit of G_N taken as $N \to \infty$. Equation (3) is the high-N approximation for the noise gain of a v.l.a. containing a finite number of sensors.

3. The high-N noise gain in anisotropic noise

The effect of the anisotropy of the ambient noise is introduced into the noise gain function in (3) through the normalized cross-spectral density Γ_κ. The second order statistical properties of plane-wave, anisotropic noise have been discussed by Cox (1973) and in particular he has considered the case of an azimuthally uniform noise field. Since the response of the v.l.a. is axially symmetrical, the system is insensitive to azimuthal variations in the noise field and therefore, for the purpose of determining the noise gain, the ambient noise may be treated as if it were azimuthally uniform. On defining the equivalent 'azimuthally uniform' directional density function as

$$\bar{F}(\theta,\bar{\omega}) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta,\phi,\bar{\omega})\,\mathrm{d}\phi, \tag{4}$$

where $F(\theta,\phi,\bar{\omega})$ is the actual directional density function of the noise field, and normalizing according to the condition

$$\frac{1}{2}\int_0^{\pi} \bar{F}(\theta,\bar{\omega})\sin\theta\,\mathrm{d}\theta = 1, \tag{5}$$

it follows immediately from Cox's analysis that

$$\Gamma_\kappa = \frac{1}{2}\int_0^{\pi} \bar{F}(\theta,\bar{\omega})\sin\theta\exp\{-\mathrm{j}k\bar{\omega}\cos\theta\}\,\mathrm{d}\theta. \tag{6}$$

Equation (6) is generally valid for arbitrary anisotropy.

Now the directional density function (hereafter used to mean $\bar{F}$ rather than F) of the noise may be expressed either as a series of zonal harmonics or, equivalently, in the form

$$\bar{F}(\theta,\bar{\omega}) = \sum_{\nu=0}^{\infty} b_\nu\cos^\nu\theta, \quad 0 \leqslant \theta \leqslant \pi, \tag{7}$$

where the coefficients b_ν are in general functions of frequency. On substituting equation (7) back into equation (6) and interchanging the order of the summation and integration, one finds that

$$\Gamma_\kappa = \frac{1}{2}\sum_{\nu=0}^{\infty} b_\nu \int_0^{\pi} \cos^\nu\theta\sin\theta\exp(-\mathrm{j}k\bar{\omega}\cos\theta)\,\mathrm{d}\theta. \tag{8}$$

Notice that the integral in (8) is real when ν is even and purely imaginary when ν is odd.

The integral, $I_{\nu,\kappa}$, in (8) satisfies the following recurrence relations:

$$I_{\nu,\kappa} = 2\frac{\sin\kappa\bar{\omega}}{\kappa\bar{\omega}} + 2\nu\frac{\cos\kappa\bar{\omega}}{\kappa^2\bar{\omega}^2} - \frac{\nu(\nu-1)}{\kappa^2\bar{\omega}^2} I_{\nu-2,\kappa}, \quad \nu \text{ even}, \tag{9a}$$

and

$$I_{\nu,\kappa} = 2\mathrm{j}\frac{\cos\kappa\bar{\omega}}{\kappa\bar{\omega}} - 2\mathrm{j}\nu\frac{\sin\kappa\bar{\omega}}{\kappa^2\bar{\omega}^2} - \frac{\nu(\nu-1)}{\kappa^2\bar{\omega}^2} I_{\nu-2,\kappa}, \quad \nu \text{ odd}. \tag{9b}$$

When these relations are combined with equations (8) and (3) to give the high-N

noise gain, an infinite number of summations over κ is obtained, each containing terms of the type

$$\frac{\sin(\kappa\bar{\omega}\alpha_{\pm})}{\kappa^n\bar{\omega}^n}, \quad n \text{ odd},$$

or of the type

$$\frac{\cos(\kappa\bar{\omega}\alpha_{\pm})}{\kappa^n\bar{\omega}^n}, \quad n \text{ even},$$

where $\alpha_{\pm} = (1 \pm \cos\theta')$.

Now, summations of this sort also appear as the Fourier expansions of the Bernoulli polynomials, whose defining equations are as follows:

$$B_1(x) = (x - \tfrac{1}{2}\sigma) = \frac{1}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin 2\pi\kappa x}{\kappa}, \quad \tfrac{1}{2}(\sigma-1) \leqslant x < \tfrac{1}{2}(\sigma+1), \tag{10a}$$

$$\frac{\mathrm{d}B_i(x)}{\mathrm{d}x} = iB_{i-1}(x), \tag{10b}$$

and

$$\int_{\frac{1}{2}(\sigma-1)}^{\frac{1}{2}(\sigma+1)} B_i(x)\,\mathrm{d}x = 0. \tag{10c}$$

The $B_i(x)$ in equations (10) are the Bernoulli polynomials themselves ($i = 1, 2, 3, \ldots$), x is the variable and σ takes the values of the odd, positive integers. Thus, according to the inequalities in (10a), when x falls in the interval (0, 1), then $\sigma = 1$; for x in (1, 2), $\sigma = 3$, etc. Evidently the Bernoulli polynomials show a discontinuous type of behaviour, with step discontinuities occurring at integer values of the variable x. The 'constant' term appearing in each of the polynomials is determined from the condition in (10c), and clearly, from the inequalities in (10a), it changes discretely as x passes through an integer value. The first five Bernoulli polynomials and their Fourier expansions are listed by Buckingham (1979) in his appendix B.

A comparison of the expression for the high-N noise gain derived from (3), (8) and (9) with the Fourier expansions of the Bernoulli polynomials leads to the following formulation† of G_∞:

$$G_\infty(\bar{\omega}, \theta') = \left[1 + \sum_{i=1}^{\infty}\frac{m_i}{\bar{\omega}^i}B_i(\tfrac{1}{2}\bar{\omega}\alpha_+/\pi) + \sum_{i=1}^{\infty}\frac{n_i}{\bar{\omega}^i}B_i(\tfrac{1}{2}\bar{\omega}\alpha_-/\pi)\right]^{-1}, \quad 0 < \theta' < \pi, \tag{11a}$$

$$G_\infty(\bar{\omega}, 0) = \left[1 + \sum_{i=1}^{\infty}\frac{m_i}{\bar{\omega}^i}B_i(\bar{\omega}/\pi) + \sum_{i=1}^{\infty}\frac{n_{2i}}{\bar{\omega}^{2i}}B_{2i}(0)\right]^{-1}, \tag{11b}$$

and

$$G_\infty(\bar{\omega}, \pi) = \left[1 + \sum_{i=1}^{\infty}\frac{m_{2i}}{\bar{\omega}^{2i}}B_{2i}(0) + \sum_{i=1}^{\infty}\frac{n_i}{\bar{\omega}^i}B_i(\bar{\omega}/\pi)\right]^{-1}, \tag{11c}$$

where the $B_{2i}(0)$ appearing in the two endfire cases are the Bernoulli numbers. The coefficients m_i and n_i are related to the coefficients b_ν, representing the anisotropy of the noise, by the relations

$$m_i = \tfrac{1}{2}(d_i - \bar{d}_i), \tag{12a}$$

$$n_i = \tfrac{1}{2}(d_i + \bar{d}_i), \tag{12b}$$

† Since writing the paper an alternative formulation of the high-N noise gain function has been suggested by a referee, for which the author is most grateful. The alternative approach is presented in appendix A.

where
$$d_i = \frac{(-2\pi)^i}{i!} \sum_{\nu=\bar{\nu}}^{\infty} \left\{ \frac{(2\nu)!}{(2\nu+1-i)!} b_{2\nu} \right\}, \tag{13a}$$

and
$$\bar{d}_i = \frac{(-2\pi)^i}{i!} \sum_{\nu=\bar{\bar{\nu}}}^{\infty} \left\{ \frac{(2\nu+1)!}{(2\nu+2-i)!} b_{2\nu+1} \right\}. \tag{13b}$$

In equation (13*a*) the lower limit on the summation is $\bar{\nu} = \frac{1}{2}i$ for i even and $\bar{\nu} = \frac{1}{2}(i-1)$ for i odd; and in (13*b*) $\bar{\bar{\nu}} = \frac{1}{2}(i-1)$ for i odd and $\bar{\bar{\nu}} = \frac{1}{2}(i-2)$ for i even.

The inequalities associated with the Bernoulli polynomials $B_i(\frac{1}{2}\bar{\omega}\alpha_+/\pi)$ and $B_i(\frac{1}{2}\bar{\omega}\alpha_-/\pi)$ in (11) are
$$\pi(\sigma_+ - 1) \leqslant \bar{\omega}\alpha_+ < \pi(\sigma_+ + 1), \tag{14a}$$

and
$$\pi(\sigma_- - 1) \leqslant \bar{\omega}\alpha_- < \pi(\sigma_- + 1), \tag{14b}$$

respectively, which are analogous to the conditions on x and σ in (10*a*). Like σ, the parameters σ_+ and σ_- are constrained to take the values of the odd, positive integers. Since the conditions in (14) give rise to discontinuities in the Bernoulli polynomials in (11), it follows that the high-N noise gain exhibits a discontinuous dependence on $\bar{\omega}$; and, bearing in mind that α_+ and α_- are independent of the anisotropy of the noise, the positions of these discontinuities depend only on the steering condition.

There is a straightforward physical interpretation of the discontinuities in the high-N noise gain. Each one occurs at a value of $\bar{\omega}$ corresponding to the appearance of an alias lobe in the beam pattern of the array. There are two sets of discontinuities, one associated with (14*a*) and the other with (14*b*), corresponding to the appearance of alias lobes in the downward and upward looking vertical directions, respectively. On the appearance of an alias lobe there is an increase in the noise observed at the output of the array and, as a result, the noise gain decreases. This happens extremely rapidly in the limit of high-N owing to the narrowness of the beams, but in a real array having a finite aperture and broader aliases, some degree of smoothing will occur to produce a less abrupt transition.

The number of non-zero coefficients in the expansion of the directional density function in (7) depends on the complexity of the noise directionality. According to the measurements of Axelrod *et al.* (1965) taken in deep water (approximately 5000 m), by using a v.l.a. moored to the bottom, the directional density function of the (downward travelling) noise, for surface wind speeds varying between Beaufort 3 and Beaufort 8 and frequencies between 112 and 1414 Hz, shows a rather slow angular dependence which could be represented satisfactorily by a sum of zonal harmonics containing terms no higher than second or third order. For such noise fields the parameters d_i and $\bar{d}_i$ in (13) are non-zero for only a few low values of i, and the number of terms contributing to the noise gain in (11) is correspondingly small. G_∞ is then particularly straightforward to evaluate. In general, if the highest order term with a non-zero coefficient in the expansion of the directional density function is $\cos^p \theta$, then the noise gain function contains Bernoulli polynomials up to and including $B_{p+1}(\ldots)$.

In an isotropic noise field, only the coefficient of the zeroth order zonal harmonic is non-zero. From the normalization condition in (5) the value of this coefficient is

$b_0 = 1$ and hence, from (13), $d_1 = -2\pi$, $d_i = 0$ for $i \geqslant 2$ and $\bar{d}_i = 0$ for all i. Therefore, from equations (11) the high-N noise gain is

$$G_\infty(\bar{\omega}, \theta') = 2\bar{\omega}/\pi(\sigma_+ + \sigma_-), \quad 0 < \theta' < \pi, \tag{15a}$$

$$G_\infty(\bar{\omega}, 0) = 2\bar{\omega}/\sigma_+\pi, \tag{15b}$$

and

$$G_\infty(\bar{\omega}, \pi) = 2\bar{\omega}/\sigma_-\pi. \tag{15c}$$

From symmetry the endfire noise gains in (15*b*) and (15*c*) are identical. The high-N noise gain in isotropic noise, from (15), is illustrated in figure 1 for three different steering conditions. For comparison, the corresponding noise gains calculated from equation (2) for an array of five elements are included in the figure. Notice that, even for such a small array, the high-N approximation is a remarkably good representation of the actual noise gain, except in the immediate vicinity of a discontinuity where the response of the real system cannot follow exactly the discontinuous curve. As N increases, of course, the fit improves until, in the limit, the two forms are indistinguishable.

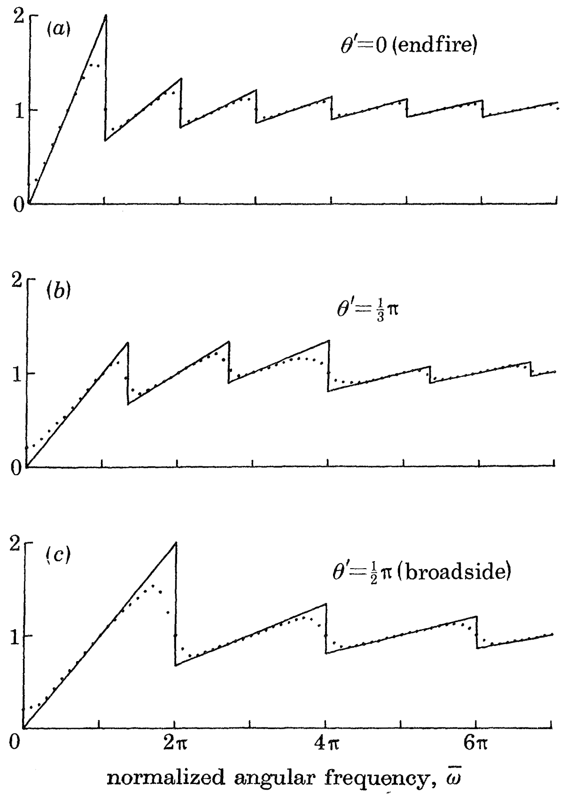

FIGURE 1. The high-N noise gain (solid line) in isotropic noise, for three different steering conditions. The dotted lines are the exact noise gains calculated from equation (2) for $N = 5$.

Figure 2 shows the high-N noise gain in an anisotropic noise field whose directional density function is

$$\bar{F}_1(\theta) = 1 + \tfrac{1}{2}\cos\theta. \tag{16}$$

The expressions for the noise gain corresponding to the three steering conditions depicted in the figure are, from equations (11),

$$G_\infty(\bar{\omega}, \tfrac{1}{3}\pi) = \frac{8\bar{\omega}^2}{\pi(\sigma_+ + \sigma_-)\,[5\bar{\omega} - (\sigma_+ - \sigma_-)\,\pi]}, \tag{17a}$$

$$G_\infty(\bar{\omega}, \tfrac{1}{2}\pi) = \bar{\omega}/\sigma\pi, \tag{17b}$$

and

$$G_\infty(\bar{\omega}, \tfrac{2}{3}\pi) = \frac{8\bar{\omega}^2}{\pi(\sigma_+ + \sigma_-)\,[3\bar{\omega} - (\sigma_+ - \sigma_-)\,\pi]}. \tag{17c}$$

Since $\sigma_+ = \sigma_-$ in the broadside array, the subscripts $\pm$ on σ have been dropped in (17b). Notice that, for the broadside case, equation (15a) is identical to (17b), indicating that the broadside array shows the same response to both isotropic noise and the noise field represented by the directional density function in (16). This is a particular example illustrating the fact that in general the broadside array shows a zero response to the odd-order zonal harmonics in the directional density function

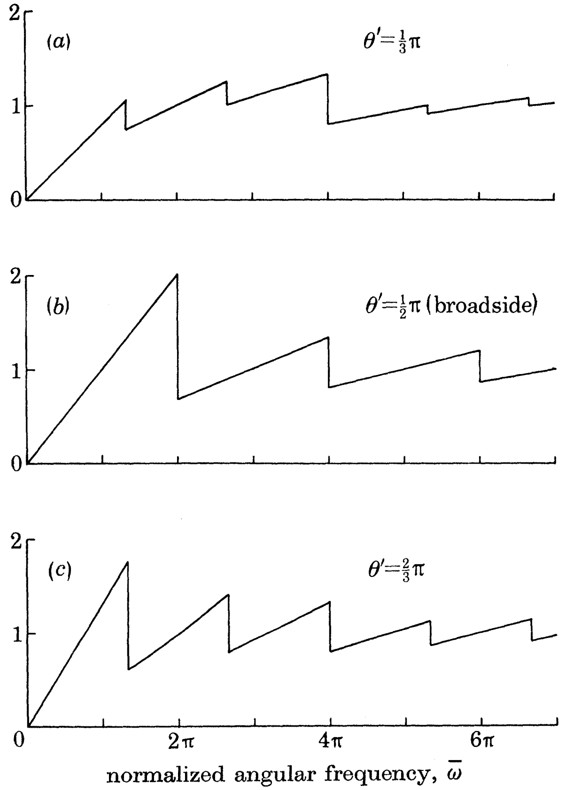

FIGURE 2. The high-N noise gain in a simple anisotropic noise field, for three different steering conditions.

of the noise. Note also that from the symmetry of the steering conditions the discontinuities in figures 2(*a*) and 2(*c*) appear at the same values of $\bar{\omega}$, but the shapes of the two curves differ as a result of the anisotropy of the noise.

For the reasons already discussed concerning the appearance of alias lobes in the beam pattern of the array, the high-N noise gain function in general displays step discontinuities, as exemplified by figures 1 and 2. However, for the particular case when the noise field shows a null in either vertical direction, no additional noise power is introduced at the output of the array when an alias appears in the direction of the null since there is then no noise power in the alias beam. Thus, in this case, the discontinuities associated with these aliases will not be steps, but discontinuities will be observed in the slope of the high-N noise gain.

4. Conclusion

It has been shown that the noise gain of a vertical line array of N equidistant elements in anisotropic noise can be approximated by representing the actual array by a notional array having the same inter-element spacing but an infinite aperture (the high-N approximation). On the basis of this approximation an analytical solution for the noise gain can be obtained which, for noise fields such as those actually observed in the deep ocean whose directional density functions can be expressed in terms of only a few low-order zonal harmonics, is relatively easy to evaluate. The main feature on the high-N noise gain curves takes the form of two sets of discontinuities, each associated with the appearance of alias lobes in the beam pattern of the array.

In practice the method presented here would probably not be used for optimizing the signal detection performance of a vertical line array, since factors such as the selection of the steering angle of the primary beam depend mainly on signal considerations. However, once the configuration of the elements in the array and the appropriate steering condition have been determined, and provided the directionality of the noise field is known, the treatment of the noise gain presented here can be usefully employed to give a very good estimate of the effect of the ambient noise on the array gain of the system.

Appendix A

The derivation of the high-N noise gain suggested by a referee is based on the identity (see, for example, Jones 1966),

$$\sum_{\kappa=1}^{\infty} \cos \kappa x = \pi \sum_{m=-\infty}^{\infty} \delta(x-2m\pi)-\tfrac{1}{2}, \tag{A 1}$$

where $\delta(\ldots)$ is the Dirac delta function. Now, from equations (3) and (6) in the text the high-N noise gain can be expressed in the form

$$G_{\infty}(\bar{\omega},\theta') = \left[1+\sum_{\kappa=1}^{\infty}\int_0^{\pi} \bar{F}(\theta)\cos\{\kappa\bar{\omega}(\cos\theta'-\cos\theta)\}\sin\theta\,\mathrm{d}\theta\right]^{-1}, \tag{A 2}$$

which, on interchanging the order of the summation and integration and using (A 1), reduces to

$$G_\infty(\bar{\omega}, \theta') = \left[\frac{\pi}{|\bar{\omega}|} \sum_{m=-M_1}^{M_2} \bar{F}'\left(\cos\theta' - \frac{2m\pi}{\bar{\omega}}\right)\right]^{-1}, \tag{A 3}$$

where the normalization condition in (5) has been used to evaluate the contribution from the term equal to $-\frac{1}{2}$ in (A 1), and $\bar{F}'(\cos\theta) = \bar{F}(\theta)$. The limits on the summation in (A 3) are

$$\left.\begin{aligned} M_1 &= \text{integer part of } (1-\cos\theta')\,\bar{\omega}/2\pi, \\ M_2 &= \text{integer part of } (1+\cos\theta')\,\bar{\omega}/2\pi, \end{aligned}\right\} \tag{A 4}$$

which, since they vary discretely with frequency, introduce discontinuities into G_∞. This is consistent with the treatment in the main text involving Bernoulli polynomials, where the discontinuities appear as a result of the discretely varying coefficients in the polynomials themselves. By comparing equations (14) and (A 4) it is easy to verify that the discontinuities in G_∞ predicted by the two analyses do indeed fall at the same values of $\bar{\omega}$.

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