Causality, Stokes' wave equation, and acoustic pulse propagation in a viscous fluid

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Stokes' acoustic wave equation is solved for the impulse response of an isotropic viscous fluid. Two exact integral forms of solution are derived, both of which are causal, predicting a zero response before the source is activated at time $t=0$. Moreover, both integral solutions satisfy a stronger causality condition: the pressure pulse is *maximally flat*, with all its time derivatives identically zero at $t=0$, signifying that there is no instantaneous response to the source anywhere in the fluid. A closed-form approximation for each of the two integrals is derived, with distinctly different properties in the two cases, even though the original integrals are equivalent in that they predict identical pulse shapes. One of these approximations, reminiscent of transient solutions that have appeared previously in the literature, is noncausal due to the incorrect representation of high-frequency components in the propagating pulse. In the second approximation, all frequency components are treated correctly, leading to an impulse response that satisfies the strong causality condition, also satisfied by the original integrals, whereby the predicted pressure pulse is zero when $t<0$ and maximally flat everywhere in the fluid immediately after $t=0$.

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I. INTRODUCTION

Acoustic propagation in a viscous fluid is characterized by a classical wave equation that was originally published by Stokes [1]. Assuming an impulsive source of strength Q at position $r=r'$, the three-dimensional, inhomogeneous form of Stokes' equation is

$$\nabla^2 g - \frac{1}{c_o^2}\frac{\partial^2 g}{\partial t^2} + \gamma\frac{\partial}{\partial t}\nabla^2 g = -Q\,\delta(\underline{r}-\underline{r'})\,\delta(t), \quad (1)$$

where $\delta(\,)$ is the Dirac δ function, g is the velocity potential, t is time, ∇^2 is the Laplacian, c_o is the speed of sound in the fluid in the absence of viscous loss, and the coefficient γ is related to the dynamic viscosity, μ, and fluid density, ρ, as follows:

$$\gamma = \frac{4\mu}{3\rho c_o^2}. \quad (2)$$

Equation (1) is a parabolic, third-order partial differential equation. Solutions of the homogeneous form of Eq. (1) for harmonic waves are well known, dating back to Stokes [1] himself, Stefan [2], Rayleigh [3], and many others. Such solutions yield the dispersion relations [4–6] for the medium, that is, the sound speed and attenuation as functions of frequency, and these expressions are consistent with the Kramers-Kronig relations, which is a necessary and sufficient condition if the transient solutions of Eq. (1) are to satisfy causality.

Transient solutions of Stokes' wave equation are less well understood than those for harmonic waves. Basically, two types of solution to the transient problem have been developed: (i) closed-form approximations [7] and (ii) solutions in the form of series [8,9] or integrals [10] suitable for numerical evaluation. As stated recently by Cobbold *et al.* [11], a difficulty with the approximate, closed-form type of solution has been, that "… approximate solutions to such problems do not satisfy causality in the strict sense, i.e., a propagated pulse does not have a sharp front but extends asymptotically to plus and minus infinity, ….". Examples of noncausal approximations for transient solutions of Stokes' equation may be found in White [4].

A more serious objection, however, has been raised by several authors [7,12], who have questioned the validity of Stokes' equation itself, with Jordan *et al.* [12] making the categorical statement that "… solutions of the classical equation of motion for this problem do not satisfy causality". If this claim were true, it would be inconsistent with the fact that the dispersion relations derived from the harmonic form of Stokes' equation satisfy Kramers-Kronig. The same authors also claim [12] that transient solutions of Stokes' equation "are felt instantly" throughout the entire fluid domain, echoing a conclusion reached earlier by Blackstock [7] and by Norwood [9]. Generally, these authors have recognized, of course, that an instantaneous response throughout the fluid is unphysical, since it implies an infinite wave speed.

The prediction of an apparently instantaneous response is not unique to Stokes' equation. As pointed out by Weyman [13], another parabolic equation, the diffusion equation, is commonly said to lead to an infinite speed of propagation through the medium supporting the motion, even though such behavior is entirely unphysical. In this case, the upper limit on the diffusion speed is expected be of the order of the speed of sound in the medium. Parabolic differential equations, it seems, even though derived correctly to represent a particular physical process, lead to unphysical results. But this is a paradoxical situation that must have a rational resolution, which is the topic of this article.

It is demonstrated below that the Green's function, or impulse response, of a viscous fluid, as derived from Stokes'

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equation, is strictly causal, with an identically zero response in the medium at all times, $t<0$, prior to the source being activated at $t=0$. Furthermore, it is shown that the instantaneous response at $t=0+$ is zero everywhere within the fluid. The latter condition follows from the fact that at the origin of time the received pulse everywhere in the fluid is perfectly smooth in the sense of being *maximally flat*: at $t=0$ *the pressure and all its time-derivatives are identically zero*. It follows that the coefficient of every term in the Taylor expansion of the pressure pulse about the origin of time is zero and hence there is no disturbance anywhere in the fluid at time $t=0+$. Since the impulse response is causal and maximally flat at the origin, it is easily shown that the same is true of any switched signal propagating through the fluid. By satisfying causality and exhibiting the maximally flat condition, the transient solutions of Stokes' equation are not only perfectly physical but also consistent with the dispersion relations.

Before developing the impulse-response solutions of Stokes' equation, the dispersion relationships are derived and their implications discussed in the context of pulse propagation. This is followed by an analysis of the impulse response from a planar source of the type considered recently by Jordan *et al.* [12]. Analogous solutions are then outlined for an infinite line source and a point source. Besides the geometrical spreading factors, the shapes of the impulse responses from the planar, line, and point sources differ, but all satisfy causality and all are maximally flat everywhere in the fluid at the origin of time.

In all three cases, the exact impulse response can be expressed in two ways, either as an inversion integral over wave number or an inversion integral over frequency. The two forms are equivalent in that they yield identical pulse shapes. From each of the inversion integrals, a closed-form, approximate solution is derived for the case of the planar source. One of these approximations is noncausal, exhibiting nonphysical properties reminiscent of those that have been found and discussed by a number of previous authors [7,9,11,12]. In contrast, the second approximation, like the inversion integrals themselves, is strictly causal and maximally flat everywhere in the fluid domain. The fundamental difference between the two approximations is that in the noncausal case, the high-frequency components of the pulse are approximated poorly, whereas the high frequencies are treated correctly in the causal approximation. The high-frequency Fourier components, of course, dictate the behavior of the pulse around the origin of time.

To conclude the discussion, the diffusion equation is briefly considered and shown to yield transient solutions with properties similar to those of Stokes' equation: the flux is strictly causal, and it is maximally flat at the origin of time, indicating no instantaneous arrivals anywhere in the medium. The essential conclusion is that Stokes' wave equation and the diffusion equation are well behaved, with transient solutions that are entirely physical for all causal driving functions.

II. DISPERSION RELATIONS

To establish the dispersion relations associated with Stokes' equation, a bilateral Fourier transform is applied to Eq. (1). The bilateral form of the transform is essential here, since this is the beginning of the investigation of causality. If any noncausal disturbance were predicted in the fluid at negative times, it would be included in the bilateral transform. In contrast, any unilateral transform presupposes that the field is strictly causal, which is the very property that is to be proved.

The bilateral Fourier transform and its inverse are

$$g_\omega(x) = \int_{-\infty}^{\infty} g(t,x)\exp(-i\omega t)dt \tag{3a}$$

$$g(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_\omega(x)\exp(i\omega t)d\omega, \tag{3b}$$

where $i=\sqrt{-1}$, ω is angular frequency, and the transform variable used as a subscript identifies the transformed field, a convenient convention when multiple transforms are employed [14]. When Eq. (3a) is applied to Eq. (1), Stokes' equation reduces to

$$\nabla^2 g_\omega + \frac{\omega^2}{c_o^2(1+i\omega\gamma)}g_\omega = -\frac{Q}{(1+i\omega\gamma)}\delta(\underline{r}-\underline{r'}), \tag{4}$$

which is the inhomogeneous form of the familiar Helmholtz equation.

In the absence of viscosity, it is obvious from Eq. (4) that the fluid is nondispersive with a sound speed equal to c_o and an attenuation coefficient of zero. When $\gamma\neq 0$, a complex sound speed $\hat{c}$ may be introduced through the expression

$$\frac{\omega}{\hat{c}} = \frac{\omega}{c} + i\alpha, \tag{5}$$

where c is the phase speed and α is the attenuation coefficient, both of which vary with frequency. By comparing Eq. (5) with the coefficient of the second term in Eq. (4), the dispersion relations for the viscous fluid may be written immediately as

$$c = \frac{c_o}{\mathrm{Re}[(1+i\omega\gamma)^{-1/2}]} = \frac{\sqrt{2}c_o\sqrt{1+\omega^2\gamma^2}}{[1+\sqrt{1+\omega^2\gamma^2}]^{1/2}} \rightarrow \begin{cases} c_o & \text{for } \omega \ll \gamma^{-1} \\ c_o\sqrt{2\omega\gamma} & \text{for } \omega \gg \gamma^{-1} \end{cases} \tag{6}$$

and

$$\alpha = -\frac{\omega\,\mathrm{Im}[\sqrt{1-i\omega\gamma}]}{c_o\sqrt{1+\omega^2\gamma^2}} = \frac{\omega}{c_o\sqrt{2}}\left\{\frac{\sqrt{1+\omega^2\gamma^2}-1}{1+\omega^2\gamma^2}\right\}^{1/2} \rightarrow \begin{cases} \dfrac{\omega^2\gamma}{2c_o} & \text{for } \omega \ll \gamma^{-1} \\ \dfrac{1}{c_o}\sqrt{\dfrac{\omega}{2\gamma}} & \text{for } \omega \gg \gamma^{-1} \end{cases}. \tag{7}$$

These expressions are plotted in dimensionless form in Fig. 1. Below the transition (angular) frequency γ^{-1}, the sound speed is essentially constant, with a value equal to c_o, while the attenuation varies as the square of the frequency. Above

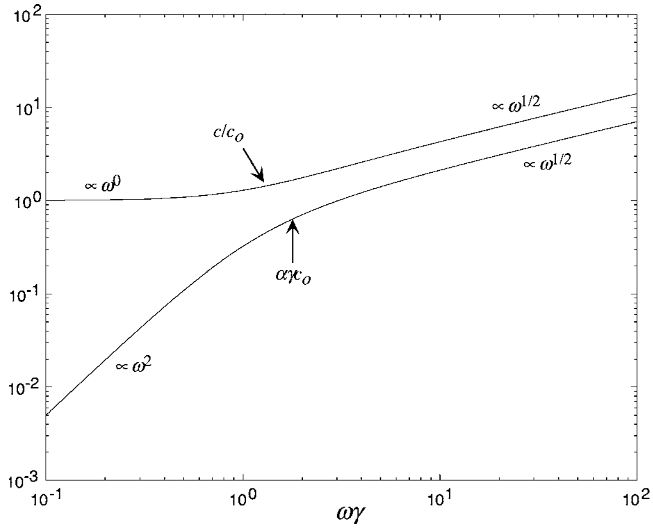

FIG. 1. Phase speed [Eq. (6)] and attenuation [Eq. (7)] as functions of frequency, all in dimensionless form.

the transition frequency, both the sound speed and the attenuation increase to indefinitely high values, although relatively slowly, as the square root of frequency. This is the classic dispersion behavior of acoustic waves in a viscous fluid.

The dispersion relations in Eqs. (6) and (7) satisfy the Kramers-Kronig relations, indicating that the field derived from the originating partial differential equation, Eq. (1), must satisfy causality. In addition, the dispersion relations prohibit an instantaneous response to the source anywhere in the fluid, a fact which may be appreciated from the following argument.

As illustrated in Fig. 1, the phase speed increases to an infinite value in the limit of high frequency, suggesting at first glance that an instantaneous response at points in the medium remote from the source might be possible. But such a conclusion would be false because, as shown in Fig. 1, any infinitely fast wave suffers an infinite attenuation and accordingly its propagation distance in the medium is zero. Such waves cannot, therefore, contribute to the field at any finite distance from the source. If there are no infinitely fast Fourier components in the propagating pulse, there can be no instantaneous response to the source anywhere in the fluid. At all finite frequencies, the wave speed and the attenuation are themselves finite and the associated Fourier components are therefore well behaved and physical. Moreover, since the highest frequencies suffer the highest attenuation, it may be anticipated that the amplitude of the transmitted pulse everywhere in the fluid will increase very smoothly after the source is activated at $t=0$.

Although this is a qualitative view of the pulse arrivals in the viscous fluid, it is representative of the physical situation. The pulses that are derived below, for three different source geometries, are strictly causal and are extremely smooth around the origin of time, with zero response at the instant immediately after the source is activated.

The solution of Eq. (1) is the Green's function, or impulse response, of the viscous fluid. Once the Green's function is known, the field generated by a source with arbitrary time dependence may be readily obtained using standard techniques of linear systems theory. To obtain the Green's function, it is necessary to apply an appropriate integral transform to the Helmholtz equation in Eq. (4), the type of transform depending on the source. To begin, a planar source is considered, of the type discussed by Jordan *et al.* [12].

III. PLANAR SOURCE

It is assumed that a planar source is located at $x=0$ and that identical, plane-wave pulses propagate away from the source in the positive and negative x directions. Adopting Cartesian coordinates, the Helmholtz equation, Eq. (4), may then be expressed in one-dimensional form as

$$\frac{\partial^2 g_\omega}{\partial x^2} + \frac{\omega^2}{c_o^2(1+i\omega\gamma)} g_\omega = -\frac{Q_p}{(1+i\omega\gamma)}\delta(x), \tag{8}$$

where Q_p has dimensions of volume per unit area.

This equation may be solved by the application of a bilateral Fourier transform over distance, x, defined as

$$g_{\omega p}(x) = \int_{-\infty}^{\infty} g_\omega(x)\exp(-ipx)dx \tag{9a}$$

with inverse

$$g_\omega(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{\omega p}\exp(ipx)dp, \tag{9b}$$

where p is the wave number. Bearing in mind the radiation condition and the familiar properties of the transforms of derivatives, when Eq. (9a) is applied to Eq. (8), the following algebraic expression for the doubly transformed field is obtained:

$$g_{\omega p} = \frac{Q_p}{p^2(1+i\omega\gamma) - k_o^2}, \tag{10}$$

where $k_o=\omega/c_o$ is the acoustic wave number. Obviously, Eq. (10) represents an exact solution of Stokes' equation, since no approximations have been introduced.

To return to the time domain, two Fourier inversions must be applied to Eq. (10), which will yield the impulse-response function. The order in which these inversions is performed is arbitary but, either way, the result is an integral that cannot be expressed explicitly. In one case, however, the integral leads to an approximation that is noncausal, whereas in the other case, the approximation for the integral is strictly causal and well behaved. The integrals themselves, of course, are equivalent and both satisfy causality, it is only the approximations that differ.

A. A noncausal approximation for the pressure pulse

It is perhaps natural to perform the inverse transform over wave number first, to obtain an expression for the frequency spectrum of the pulse

$$g_\omega = \frac{Q_p}{2\pi(1+i\omega\gamma)}\int_{-\infty}^{\infty}\frac{\exp(ipx)}{(p-p_+)(p-p_-)}dp, \tag{11}$$

where the simple poles $p_\pm$ are the roots of the quadratic denominator in Eq. (10), which are given by the expression

$$p_{\pm} = \pm \frac{k_o}{\sqrt{1 + i\omega\gamma}}. \tag{12}$$

For real (positive or negative) ω, the poles p_+ and p_-, respectively, lie in the bottom half and top half of the complex p plane. To evaluate the integral in Eq. (11), the appropriate contour of integration is around the top (bottom) half plane for $x > 0$ ($x < 0$), which leads to the following exact expression for the spectrum of the velocity potential [4]:

$$g_{\omega} = \frac{Q_p}{2ik_o\sqrt{1 + i\omega\gamma}} \exp\left(-\frac{ik_o|x|}{\sqrt{1 + i\omega\gamma}}\right). \tag{13}$$

Converting from velocity potential $g(t,x)$ to pressure $p(t,x)$, the Fourier inversion over frequency leads to the exact integral representation

$$p(t,x) = \rho\frac{dg}{dt} = \frac{\rho c_o Q_p}{4\pi}\int_{-\infty}^{\infty} (1 + i\omega\gamma)^{-1/2} \times \exp\left(-\frac{ik_o|x|}{\sqrt{1 + i\omega\gamma}}\right)\exp(i\omega t)d\omega. \tag{14}$$

The integral in Eq. (14) cannot be evaluated explicitly but an approximate form may be obtained by expanding the radical in a Taylor series, as follows:

$$(1 + i\omega\gamma)^{-1/2} = 1 - \frac{i\omega\gamma}{2} + \cdots. \tag{15}$$

The expression for the pressure then becomes

$$p(t,x) \approx \frac{\rho c_o Q}{4\pi}\int_{-\infty}^{\infty} \exp\left[i\omega\left(t - \frac{|x|}{c_o}\right)\right]\exp\left(-\frac{\omega^2\gamma|x|}{2c_o}\right)d\omega, \tag{16}$$

where the terms up to first order in $\omega\gamma$ in the expansion of the radical in Eq. (15) have been included in the argument of the exponential but elsewhere the radical has been set equal to unity. With these approximations, the integral for the pressure pulse has been reduced to a known form [15], allowing the solution to be expressed as

$$p(t,x) \approx \frac{\rho c_o Q}{2}\sqrt{\frac{c_o}{2\pi\gamma|x|}}\exp\left[-\frac{(c_o t - |x|)^2}{2c_o\gamma|x|}\right]. \tag{17}$$

Notice that this is not a retarded potential solution, since it depends on t and x individually and not on $(c_o t - x)$ alone. However, in the limit of zero viscosity (i.e., $\gamma \to 0$), Eq. (17) reduces to the correct retarded-potential form

$$\lim_{\gamma\to 0} p(t,x) = \frac{\rho c_o Q}{2}\delta\left(t - \frac{|x|}{c_o}\right). \tag{18}$$

This limit was derived using a well-known δ-function identity [16].

An expression equivalent to Eq. (17) but for particle displacement has been discussed by White [4]. The Gaussian form exhibited by Eq. (17) is reminiscent of Blackstock's [7] approximate expression for the behavior of a switched sinusoid in a viscous fluid, although the details of the two solutions are distinct since they represent different types of transient signals. All such approximate solutions suffer from the same problem: they fail to satisfy causality. This difficulty is apparent in Eq. (17), which predicts a finite pressure everywhere in the viscous fluid for nonpositive times. Although several authors have suggested otherwise [7,12], such noncausal behavior is not due to a failure of Stokes' viscous wave equation [Eq. (1)]. In the present case, the problem is a consequence of an unsatisfactory approximation, Eq. (15), which misrepresents the high frequencies in the pulse, hence leading to the unphysical solution for the impulse response in Eq. (17).

B. A causal approximation for the pressure pulse

Equation (17) fails to predict the correct behavior not only for negative times but also at $t=0$ and immediately thereafter. In particular, Eq. (17) indicates (erroneously) that a signal appears everywhere throughout the viscous fluid at the instant the source is activated. Such an unphysical prediction is a consequence of the poorly represented high frequencies in the pulse, arising from the approximation in Eq. (15). To derive a causal approximation for the pressure pulse, an alternative approach must be adopted in which Eq. (15) is abandoned and all frequencies are treated correctly.

To this end, consider again the exact algebraic expression for the doubly transformed velocity potential in Eq. (10). Now, the inverse Fourier transform over *frequency* is performed first, to return the wave number spectrum

$$g_p = -\frac{Q_p c_o^2}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(i\omega t)}{(\omega - \omega_+)(\omega - \omega_-)}d\omega, \tag{19}$$

where the simple poles $\omega_{\pm}$ are the roots of the quadratic denominator in Eq. (10):

$$\omega_{\pm} = \frac{i\gamma c_o^2 p^2 \pm \sqrt{4c_o^2 p^2 - \gamma^2 c_o^4 p^4}}{2}. \tag{20}$$

With p real, both poles lie above the real axis in the top half of the complex ω plane. From Jordan's lemma and Cauchy's theorem, the D-shaped integration contour used in evaluating Eq. (19) for negative times must be taken around the lower half plane. Since this contour encloses no poles, it follows immediately that the field for $t < 0$ is identically zero. For positive times, the integration contour must be taken around the top half plane, yielding a nonzero result since the contour encloses the two poles in Eq. (20). By adding the residues of these poles, the wave-number spectrum of the field can be written exactly for all times t as

$$g_p = u(t)\frac{2Q_p c_o^2}{\sqrt{4c_o^2 p^2 - \gamma^2 c_o^4 p^4}}\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right) \times \sin\left(\frac{\sqrt{4c_o^2 p^2 - \gamma^2 c_o^4 p^4}}{2}t\right), \tag{21}$$

where $u(t)$ is the Heaviside unit step function.

Obviously, Eq. (21) satisfies causality, since it is identically zero for all negative times. As every wave number component of the field is strictly causal, it follows that the field itself must also be causal. To obtain the Green's function for the field, the inverse Fourier transform with respect to wave number p is applied to Eq. (21):

$$g(t,x) = u(t)\frac{Q_p c_o}{2\pi}\int_{-\infty}^{\infty}\frac{1}{p\chi}\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\times\sin(c_o p\chi t)\exp(ipx)dp, \tag{22}$$

where

$$\chi = \sqrt{1-\frac{\gamma^2 c_o^2 p^2}{4}}. \tag{23}$$

By differentiating Eq. (22) with respect to t, the exact solution for the pressure pulse is found to be

$$p(t,x) = \rho\frac{dg}{dt} = u(t)\frac{\rho c_o^2 Q_p}{4\pi}\int_{-\infty}^{\infty} e^{ipx}\left[\left(1+\frac{i\gamma c_o p}{2\chi}\right)\exp\left\{\left(ic_o p\chi - \frac{\gamma c_o^2 p^2}{2}\right)t\right\} + \left(1-\frac{i\gamma c_o p}{2\chi}\right)\exp\left\{-\left(ic_o p\chi + \frac{\gamma c_o^2 p^2}{2}\right)t\right\}\right]dp. \tag{24}$$

Although Eq. (24) is obviously causal, there is a stronger condition on the field at the origin of time that emerges from the integral formulation of Eq. (24): at $t=0$, *the pressure and all its time-derivatives are identically zero at every point in the viscous medium*. This is proved by taking the n^{th} time derivative under the integral sign and then setting $t=0$. A sum of terms is obtained, each one of which is an even moment of the function $\cos(px)$, that is, each of the terms is an integral of the form

$$m_{2q} = \int_0^{\infty} p^{2q}\cos(px)dp, \tag{25}$$

where q is a non-negative integer. Since the zeroth moment is

$$m_o = \int_0^{\infty}\cos(px)dp = \pi\delta(x), \tag{26}$$

all the higher, even moments may be expressed as even-order derivatives of the delta function

$$m_{2q} = (-1)^q\frac{d^{2q}m_o}{dx^{2q}} = (-1)^q\pi\delta^{(2q)}(x), \tag{27}$$

where the superscript on the δ function denotes the $(2q)$ derivative with respect to x. Thus, all the moments represented by Eq. (25) exhibit a singularity at the origin of x (the source position) and are precisely zero everywhere else throughout the fluid. It follows that the field expression in Eq. (24) satisfies causality in the strong sense that, at the origin of time when the source is activated, $t=0$, the pressure is *maximally flat* everywhere throughout the viscous fluid: not only is the pressure identically zero at $t=0+$ but so too are all the time derivatives of the pressure pulse. The Taylor expansion of the pressure taken around the origin of time is therefore zero and hence there are no instantaneous arrivals predicted anywhere in the fluid.

Returning now to the evaluation of Eq. (24), the integrals cannot be expressed explicitly but, on approximating the radical in Eq. (23) as unity, the impulse response reduces to

$$p(t,x) \approx u(t)\frac{\rho Q_p c_o^2}{4\pi}\int_{-\infty}^{\infty}\left\{\left(1+\frac{i\gamma p c_o}{2}\right)\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\exp[ip(x+c_o t)] + \left(1-\frac{i\gamma p c_o}{2}\right)\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\exp[ip(x-c_o t)]\right\}dp. \tag{28}$$

The integrals here are known forms [17], allowing the pressure to be approximated as

$$p(t,x) \approx u(t)\frac{\rho c_o Q_p}{4\sqrt{2\pi\gamma t}}\{F(t,x)+F(t,-x)\}, \tag{29a}$$

where

$$F(t,x) = \left(1+\frac{x}{c_o t}\right)\exp\left[-\frac{(c_o t - x)^2}{2\gamma c_o^2 t}\right]. \tag{29b}$$

In common with the noncausal approximation for the pressure in Eq. (17), the pressure pulse in Eqs. (29a) and (29b) is not a retarded potential, since it depends on x and t

independently and not on $(c_o t - x)$ alone. And like Eq. (17), in the limit of zero viscosity, Eq. (29a) reduces to

$$\lim_{\gamma\to 0} p(t,x) = \frac{\rho c_o Q_p}{2}\delta\left(t - \frac{|x|}{c_o}\right), \tag{30}$$

which has been derived with the aid of a well-known δ function identity. Equation (30) is the correct, retarded-potential solution for an impulse of pressure propagating through an inviscid fluid.

As with the exact, integral expression for the impulse response in Eq. (24), the approximation in Eqs. (29) is strictly causal in the strong sense of being maximally flat everywhere throughout the viscous fluid at the origin of time, $t=0$. This is readily shown to be true, since, in the limit of small (positive) time, all the time derivatives of the pressure in Eqs. (29) are dominated by the function $\exp[-x^2/(2\gamma c_o^2 t)] \to 0$ for $x \neq 0$. Thus, everywhere in the viscous fluid, the pressure makes a perfectly smooth, flat transition from zero at negative and zero times to finite values as t becomes positive. Such behavior is entirely physical, with no arrivals anywhere in the fluid at the instant the source begins to transmit, which is consistent with the dispersion relations in Eqs. (6) and (7). There are arrivals, however, before the retarded time $t_o = |\mathrm{x}|/c_o$, as expected, since, according to Eq. (6), the phase speed increases indefinitely with increasing frequency, albeit slowly, as $\omega^{1/2}$.

C. Comparison of impulse-response solutions

To compare the noncausal approximation [Eq. (17)], the causal approximation [Eqs. (29)] and the exact integrals [Eq. (14) and (24)] for the pressure pulse in a viscous fluid, it is convenient to introduce a normalization scheme based on the retarded time $t_o = |\mathrm{x}|/c_o$. The normalizing pressure is taken to be the peak value of the noncausal, symmetrical Gaussian pulse in Eq. (17), which coincides with the retarded time t_o:

$$p(t_o) = \frac{\rho c_o Q_p}{2\sqrt{2\pi\gamma t_o}}. \tag{31}$$

Since γ has dimensions of time, the appropriate normalization is

$$\bar{\gamma} = \frac{\gamma}{t_o}, \tag{32}$$

where the convention of using an overbar to denote a normalized quantity has been introduced.

The normalized pressures may now be expressed as functions of normalized time, $\bar{t} = t/t_o$. Then the noncausal approximation in Eq. (17) reduces to

$$\bar{p}_{nc}(\bar{t}) \approx \exp\left[-\frac{(\bar{t}-1)^2}{2\bar{\gamma}}\right]; \tag{33}$$

and the causal approximation in Eqs. (29) becomes

$$\bar{p}_c(\bar{t}) \approx \frac{1}{2\sqrt{\bar{t}}}\{f(\bar{t}) + f(-\bar{t})\}, \tag{34a}$$

where

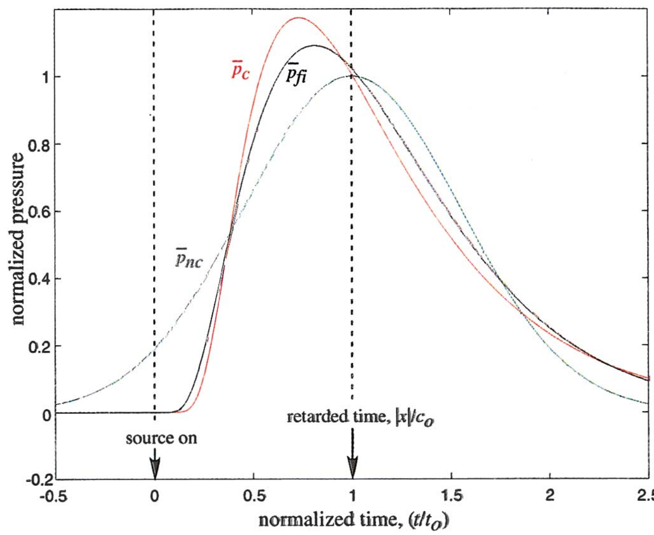

FIG. 2. (Color) Impulse response of a viscous fluid, as predicted by a numerical integration of the exact expression in Eq. (35) (black line), by the causal approximation in Eqs. (34a) and (34b) (red line), and by the noncausal approximation in Eq. (33) (green line), with $\bar{\gamma}=0.3$ in all three cases. Notice that the (causal) red and black curves are maximally flat at the origin of time.

$$f(\bar{t}) = \left(1 + \frac{1}{\bar{t}}\right)\exp\left[-\frac{(\bar{t}-1)^2}{2\bar{\gamma}\bar{t}}\right]. \tag{34b}$$

In Eqs. (33) and (34), the subscripts *nc* and *c* denote "noncausal" and "causal," respectively. The same normalization scheme may also be applied to the (exact) integrals for the pressure pulse in Eqs. (14) and (24). Since these two integral formulations yield identical results, only the simpler of the two, Eq. (14), is considered here. This reduces to the normalized form

$$\bar{p}_{fi}(\bar{t}) = \sqrt{\frac{\bar{\gamma}}{2\pi}}\int_{-\infty}^{\infty}(1+i\bar{\omega}\bar{\gamma})^{-1/2}\exp\left[i\bar{\omega}\left(\bar{t} - \frac{1}{\sqrt{1+i\bar{\omega}\bar{\gamma}}}\right)\right]d\bar{\omega}, \tag{35}$$

where $\bar{\omega} = \omega t_o$, and the subscript *fi* denotes "frequency integral."

Notice that the propagation distance x no longer appears explicitly in the normalized expressions for the pressure but instead is embedded in the normalized parameter $\bar{\gamma}$ and the normalized time $\bar{t}$. The effect of the normalization is to reduce the expressions for the pressure to functions of a single variable $\bar{t}$, involving just one parameter $\bar{\gamma}$. Under this scheme, if the distance x and viscosity μ are scaled by the same factor, $\bar{\gamma}$ stays the same and hence the shape of the normalized pressure pulse remains unchanged.

Figure 2 shows the dimensionless pressure pulses in Eqs. (33)–(35) for a value of $\bar{\gamma}=0.3$, which is unrealistically high for the experimental conditions found in most fluids but serves to illustrate the differences between the two approximations and the exact solution. The causal approximation [Eqs. (34)] and the exact solution [Eq. (35)] match reasonably well and both are well behaved around the origin of

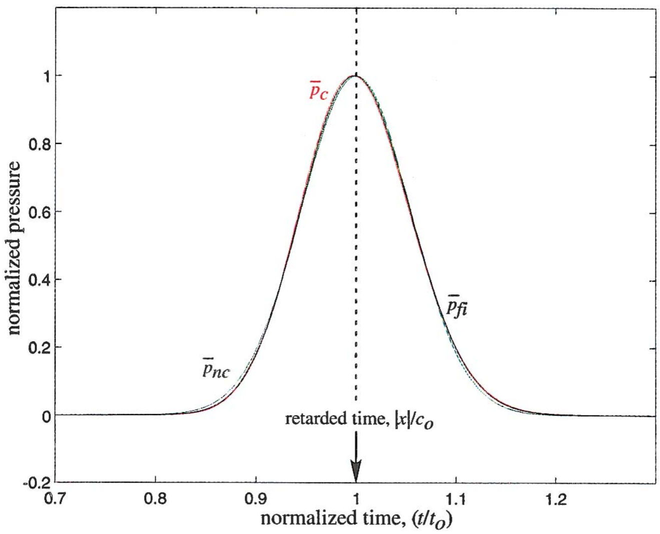

FIG. 3. (Color) Impulse response of a viscous fluid, as predicted by a numerical integration of the exact expression in Eq. (35) (black line), by the causal approximation in Eqs. (34a) and (34b) (red line), and by the noncausal approximation in Eq. (33) (green line), with $\bar{\gamma}=0.003$ in all three cases.

time, showing zero responses for $t<0$ and no arrivals (maximally flat behavior) at $t=0+$, the instant immediately after the source begins to transmit. Both pulses are asymmetrical about the peak, each exhibits a leading edge that is considerably steeper than the trailing edge, and the two peaks arrive in advance of the retarded time t_o. In contrast, the symmetrical (Gaussian) peak of the noncausal approximation arrives relatively late, at exactly the retarded time t_o, and the tail of the leading edge extends into negative times, hence the failure to satisfy causality.

The differences between the three curves in Fig. 2 become progressively less pronounced as the viscosity is reduced, as is illustrated in Fig. 3 for the case $\bar{\gamma}=0.003$. Of course, the Gaussian pulse in Eq. (33) still fails to satisfy causality, while the curves from Eqs. (34) and (35) remain strictly causal in the strong sense, but the differences between the shapes of the three curves are now almost imperceptible. Notice that the skewness, such a prominent feature of the two causal pulses in Fig. 2, has all but vanished in Fig. 3, where all three pulses are seen to be essentially symmetrical about the retarded time t_o. Indeed, as $\bar{\gamma}$ is reduced further, all three pulses approach the symmetrical, delta-function form of Eqs. (18) and (30).

IV. INFINITE LINE SOURCE

For a line source, cylindrical coordinates are appropriate, with the axis coincident with the source itself. As with the planar source, the field is one dimensional, in this case depending only on the radial distance r; and the Helmholtz equation, Eq. (4), takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g_\omega}{\partial r}\right)+\frac{\omega^2}{c_o^2(1+i\omega\gamma)}g_\omega=-\frac{Q_L}{(1+i\omega\gamma)}\frac{\delta(r)}{\pi r}, \quad (36)$$

where Q_L is the source strength with dimensions of volume per unit length. The solution for the field is similar to that for the planar source except that, instead of a bilateral Fourier transform, a Hankel transform of zero order is applied to the Helmholtz equation.

On performing the inverse transform over frequency first, the inversion integral is found to be exactly the same as that in Eqs. (19) and (20), which has already been evaluated exactly. Taking this result and performing the Hankel inversion, the following expression for the pressure pulse is obtained:

$$\begin{aligned}p(t,x)=\rho\frac{dg}{dt}=u(t)\frac{\rho c_0^2 Q_L}{4\pi}\int_0^\infty pJ_o(pr)\Bigg[\left(1+\frac{i\gamma c_o p}{2\chi}\right)\\ \times\exp\left\{\left(ic_o p\chi-\frac{\gamma c_o^2 p^2}{2}\right)t\right\}+\left(1-\frac{i\gamma c_o p}{2\chi}\right)\\ \times\exp\left\{-\left(ic_o p\chi+\frac{\gamma c_o^2 p^2}{2}\right)t\right\}\Bigg]dp, \quad (37)\end{aligned}$$

where the integration variable p is now the radial wave number, $J_o(..)$ is the Bessel function of the first kind of order zero, and χ is as defined in Eq. (23). From the presence of the Heaviside unit step function in this expression, it is clear that the pressure pulse is strictly causal, with zero response prior to $t=0$, the time at which the source is activated.

Immediately afterwards, at $t=0+$, the pressure everywhere in the fluid is zero and so too are all its time derivatives. This may be proved from a similar argument to that for the planar source. Thus, around the origin of time, the cylindrical pulse is maximally flat, its Taylor expansion is identically zero, and hence there are no instantaneous arrivals anywhere in the fluid.

V. POINT SOURCE

Again the field is one dimensional, depending only on the radial distance r from the source. The Helmholtz equation, Eq. (4), in spherical polar coordinates is therefore

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_\omega}{\partial r}\right)+\frac{\omega^2}{c_o^2(1+i\omega\gamma)}g_\omega=-\frac{Q}{(1+i\omega\gamma)}\frac{\delta(r)}{2\pi r^2}, \quad (38)$$

where Q is the source strength with dimensions of volume. By making the substitution

$$g_\omega=\frac{\phi_\omega}{\sqrt{r}}, \quad (39)$$

Eq. (38) reduces to Bessel's equation, which is solved by applying a Hankel transform of order one-half. Then, proceeding much as with the line source, the pressure is eventually found to be

$$\begin{aligned}p(t,x)=\rho\frac{dg}{dt}=u(t)\frac{2\rho c_o^2 Q}{\pi r}\int_0^\infty\frac{\sin(pr)}{\chi}\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\\ \times\left\{\frac{\gamma p^2 c_o^2}{2}\sin(p\gamma c_o t)+p\chi c_o\cos(p\chi c_o t)\right\}dp, \quad (40)\end{aligned}$$

where p is the radial wave number and χ is as defined in Eq. (23).

Clearly, the pressure pulse in Eq. (40) is causal, since it is zero for negative times. As with the pulses from the planar and line sources, all the time derivatives of Eq. (40) at $t=0$ are zero, hence the pulse is maximally flat around $t=0$, the Taylor expansion is zero for $t=0+$ and there are no instantaneous arrivals anywhere in the fluid.

VI. TRANSIENT SIGNALS IN GENERAL

Knowing the impulse response, the velocity potential $\phi(t,x)$, due to any source function $s(t)$, may be determined almost immediately. In terms of Fourier transforms with respect to time, it is almost self-evident from the Helmholtz equation [Eq. (4)] that the velocity potential is the product of the impulse response and the source function

$$\phi_\omega(\underline{r}) = s_\omega(\underline{r}')g_\omega(\underline{r}). \tag{41}$$

On converting to the time domain, this product becomes a convolution between the source function and the Green's function

$$\phi(t,\underline{r}) = \int_{-\infty}^{\infty} s(\tau,\underline{r}')g(t-\tau,\underline{r})d\tau. \tag{42}$$

The n^{th} time derivative of the velocity potential is therefore

$$\frac{d^n\phi}{dt^n} = \int_{-\infty}^{\infty} s(\tau,\underline{r}')\frac{d^n}{dt^n}g(t-\tau,\underline{r})d\tau, \tag{43}$$

which returns the pressure when $n=1$ and all the time derivatives of the pressure when $n\geqslant 2$.

Assuming that the source is causal, such that $s(t,\underline{r}')=0$ for $t<0$, the lower limit on the integral in Eq. (43) may be replaced by zero. Moreover, since it has been proved above that all the time derivatives of the impulse-response function are zero for $t\leqslant 0$, the upper limit on the integral in Eq. (43) may be replaced by t. Thus, the convolution for the time derivatives of the velocity potential becomes

$$\frac{d^n\phi}{dt^n} = \int_{0}^{t} s(\tau,\underline{r}')\frac{d^n}{dt^n}g(t-\tau,\underline{r})d\tau. \tag{44}$$

At $t=0$, the instant the source activates, it is clear from Eq. (44) that the pressure $(n=1)$ and all its time derivatives $(n\geqslant 2)$ are identically zero, regardless of the shape of the source function.

Evidently, for any causal source function, the solution of Stokes' wave equation [Eq. (1)] for the acoustic field in a viscous fluid is a transient (possibly followed by a steady-state disturbance, for instance, in the case of a switched sinusoid) that satisfies causality in the strong sense, that is to say, the pressure is maximally flat at the origin of time $t=0$. Although several authors [7,9,11,12] have stated otherwise, this precludes the possibility that any acoustic disturbance is felt throughout the medium at the instant the source is activated. Stokes' wave equation yields perfectly physical solutions for the transient pressure in a viscous fluid, regardless of the detailed shape of the causal source function.

VII. THE DIFFUSION EQUATION

The diffusion equation is, perhaps, a simpler example of a parabolic partial differential equation than Stokes' wave equation. With regard to causality, the solutions of the two equations have several important properties in common. To illustrate the causal nature of the solutions of the diffusion equation, a simple one-dimensional problem is briefly considered below.

Suppose a planar source of fluid, in the $x=0$ plane, is injected instantaneously into an isotropic host fluid at time $t=0$. If, after time t at distance x from the source, the concentration of introduced fluid is $\beta(t,x)$, then β is a solution of the inhomogeneous diffusion equation [18]

$$D\frac{\partial^2\beta}{\partial x^2} - \frac{\partial\beta}{\partial t} = -q\delta(x)\delta(t), \tag{45}$$

where D is the diffusion coefficient and q is the source "strength" with dimensions of concentration times length.

Equation (45) may be solved for the "pulse" of concentrate diffusing through the host fluid by using similar integral transform techniques to those applied earlier to Stokes' equation. In the present case, the two transforms in question are both bilateral Fourier transforms, taken over time and distance, which lead to the following doubly transformed expression for the concentration:

$$\beta_{\omega p} = \frac{q}{(Dp^2 + i\omega)}, \tag{46}$$

where ω and p are the integration variables in the temporal and spatial Fourier transforms, respectively. In the complex frequency plane, and for p real, Eq. (46) has a simple pole on the imaginary axis at $\omega_o = +iDp^2$. Therefore, on performing the inverse Fourier transform with respect to frequency, taking a D contour in the top half plane for $t>0$ and in the lower half plane for $t<0$, the solution for the spatial spectrum of the concentration pulse is immediately found to be

$$\beta_p = u(t)qe^{-Dp^2t}. \tag{47}$$

The presence of the Heaviside unit step function here signifies that every spatial Fourier component is strictly causal, exhibiting a zero response for all negative times. It follows that the pulse itself must also satisfy causality.

The solution for β is now obtained by applying the inverse spatial Fourier transform to Eq. (47), which yields

$$\beta(t,x) = u(t)\frac{q}{4\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{48}$$

This familiar, unimodal expression for the concentration as a function of time and distance from the source exhibits a maximum when

$$t = t_m = \frac{x^2}{2D}. \tag{49}$$

The leading edge of this peak does not rise up from zero in a discontinuous fashion, since all the time derivatives of β have as a dominant factor the exponential function in Eq. (48). Hence, at the origin of time, every time derivative of β

is zero. Thus, like the acoustic pulses treated earlier, the expression for the concentration in Eq. (48) is perfectly smooth in the sense of being maximally flat at the origin of time, $t=0$. It follows that the Taylor expansion of the pulse about the origin of time is zero, indicating that there is no instantaneous arrival anywhere in the host fluid, that is, $\beta(0+,x)=0$. As its solutions satisfy this strong causality condition, the essential conclusion is that the diffusion equation provides a perfectly physical representation of diffusive processes.

VIII. CONCLUDING REMARKS

Since 1845, when Stokes [1] introduced his acoustic wave equation for a viscous fluid, its solutions for pulse propagation have proved to be elusive. Over recent years, it appears to have become accepted that Stokes' wave equation leads to predictions of noncausal propagation in the form of acoustic arrivals everywhere throughout the fluid at the instant the source is activated. Such behavior would imply an infinite speed of wave propagation, which is unphysical.

In this article, the Green's function, or impulse response, of a viscous fluid is derived from Stokes' wave equation for the cases of a planar, linear, and point source. These exact solutions are not only zero for negative times but also satisfy a stronger causality condition: everywhere in the fluid, the predicted pressure pulse is *maximally flat* at the instant the source is activated, that is to say, the pressure and all its time derivatives are identically zero at the origin of time. It is demonstrated that this strong causality condition is satisfied not only by the impulse response but by any (exact) transient solution of Stokes' wave equation.

The condition of strong causality ensures that no acoustic disturbances are felt anywhere in the viscous fluid at the instant the source is triggered. It follows that all transient solutions of Stokes' equation are perfectly physical in their behavior and, in particular, no infinite wave speed is implied. This is consistent with the dispersion relations, which indicate that in the limit of high frequency, although the phase speed does in fact diverge, the attenuation also becomes indefinitely high, thus completely suppressing the infinitely fast, nonphysical Fourier components in the acoustic field.

In a brief discussion of the diffusion equation, the condition of strong causality is also shown to hold. Thus, the solution for a diffusive pulse and all its time derivatives are zero at the origin of time: at the instant the source is activated, the pulse is perfectly smooth, in the sense of being maximally flat, everywhere in the host fluid, with no instantaneous arrivals anywhere. As with Stokes' wave equation, it is concluded that the diffusion equation provides a valid, physical representation of transient processes in the host medium.

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