Spatial coherence and cross correlation of three-dimensional ambient noise fields in the ocean

Shane C. Walker and Michael J. Buckingham[a)]
Marine Physical Laboratory, Scripps Institution of Oceanography, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0238

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Ambient acoustic noise fields in the ocean are generally three dimensional in that they exhibit vertical and horizontal directivity. A model of spatially homogeneous noise is introduced in which the directionality is treated as separable, that is, the overall directionality of the field is the product of the individual directivities in the horizontal and vertical. A uni-modal von Mises circular distribution from directional statistics is taken to represent the noise in the horizontal, whilst the vertical component is consistent with a surface distribution of vertical dipoles. An analysis of the coherence and cross correlation of the noise at two horizontally aligned sensors is developed. The coherence function involves a single integral over finite limits, whilst the cross-correlation function, derived on the assumption that the noise has been pre-whitened, is given by an integral with limits that depend on the correlation delay time. Although the cross-correlation function does not exhibit delta functions that could be identified with the Green's function for propagation between the two sensors in the field, it does drop abruptly to zero at numerical time delays equal to the travel time between the sensors. Hence the noise could be used to recover the sound speed in the medium.
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I. INTRODUCTION

It is well established that ambient noise in the ocean can be inverted to return information about the seabed,[1–4] the water column,[5,6] and the sea surface.[7–12] Ambient noise is also a factor in determining the effectiveness of underwater acoustic signal detection systems; and the noise can be used as a source of acoustic illumination for producing images of objects located in the ocean.[13,14] A more recent application is noise interferometry, whereby the (deterministic) Green's function for acoustic propagation between two points in the ocean is recovered from the (stochastic) cross-correlation function of the noise fluctuations observed at the same two points.[15,16]

In many of these applications, the directionality of the ambient noise significantly affects the overall performance of the system or technique in question. Most signal processing algorithms, however, involve, not the directionality itself, but instead the spatial coherence or, in the case of noise interferometry, the cross-correlation function of the noise. The coherence function and cross-correlation function are related to the directionality via various types of inversion integral.

In an early model of deep-water ambient noise developed by Cron and Sherman,[17,18] the ocean is treated as a semi-infinite half space with a uniform sound speed profile, and the noise field is generated by a sheet of uncorrelated point sources that are Poisson-distributed in a horizontal plane located immediately beneath the (planar) pressure-release sea surface. In effect, each of these monopole sources combined with its negative image in the sea surface acts as a vertical dipole. Since the absence of a seabed eliminates bottom reflections, all the noise travels downward; and, from symmetry, the noise field is uniform in azimuth. An analysis of the spatial coherence and cross correlation of a selection of azimuthally uniform noise fields, including Cron and Sherman's,[17,18] has recently been presented by Buckingham.[19]

Ambient noise in the real ocean is not usually uniform in azimuth, due to the presence of localized acoustic sources such as surface shipping, storm systems, ice cover, and underwater seismic events. The purpose of this article is to introduce a model of azimuthally non-uniform noise that is representative of a localized storm system on the sea surface. In the vertical, the directionality of the noise is the same as that introduced by Cron and Sherman[17,18] for a surface sheet of uncorrelated noise sources, while the horizontal directionality is represented by a circular distribution from directional statistics known as a von Mises distribution.[20]

Although azimuthal non-uniformity has no effect on the coherence and cross correlation of the noise fluctuations at vertically aligned sensors, it can be significant for other orientations of the sensor pair. In the analysis presented below, it is assumed that the sensors are aligned horizontally, an arrangement that is commonly encountered in practice. To begin the discussion, the directional density function of the three-dimensional noise field is constructed, from which the horizontal coherence and cross correlation are subsequently derived.

II. THE DIRECTIONAL DENSITY FUNCTION

Ambient noise in the deep ocean often consists of a random superposition of uncorrelated plane waves propagating in all directions.[21] Such noise fields are spatially

[a)]Author to whom correspondence should be addressed. Electronic mail: mbuckingham@ucsd.edu

homogeneous, that is, their statistical measures, including the power spectrum, cross-spectral density, and cross-correlation function, are independent of the position at which they are observed. The noise field considered hereinafter is a superposition of two spatially homogeneous components, one from wave breaking, which occurs across the whole sea surface, and the second from a localized storm system laterally offset from the receiving station, as shown in Fig. 1. In both cases, the noise sources are represented as a plane of uncorrelated monopoles lying immediately beneath the pressure-release sea surface, which forms negative images of the monopoles to create, in effect, a surface layer of dipoles, as in the Cron and Sherman model.[17,18] Now, however, due to the limited areal coverage of the storm, the noise field may vary strongly in azimuth.

The directionality of the noise, that is, the noise power per unit solid angle, is represented by a dimensionless directional density function, $F(\theta,\phi)$, where θ and ϕ are angular coordinates in a polar coordinate system whose origin is at the midpoint between the two sensors. The polar angle, $0 \le \theta \le \pi$, is measured from the zenith and the azimuthal angle, $0 \le \phi \le 2\pi$, from the horizontal line joining the sensors. Since the noise fields generated by wave breaking and the storm system are both broad band with similar spectra, $F(\theta,\phi)$ is taken to be independent of frequency. The directional density function is normalized according to the condition[21]

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi = 1, \tag{1}$$

which, if the noise were isotropic, would yield $F = 1$, since the solid angle of a sphere is equal to 4π. In the present case,

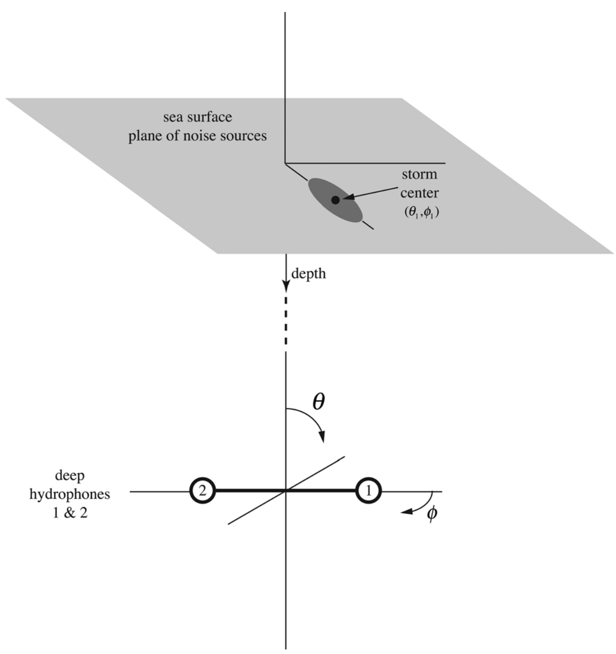

FIG. 1. Sketch showing the horizontal alignment of the two sensors, along with the polar coordinate system used to characterize the directional density function. The plane of breaking-wave sources is shown in light gray, with the region occupied by the storm sources depicted in darker gray. The angular coordinates of the storm center are (θ_1, ϕ_1). (Surface features and hydrophones not to scale.)

$F(\theta,\phi)$ is not a constant but involves two unknown coefficients, associated with the relative contributions to the total noise field from the wave-breaking and storm sources, and the normalization condition in Eq. (1) provides a relationship between them.

To proceed further, the directionality of the noise must be specified. Since both the wave sources and the storm sources may be regarded as surface dipoles, with a higher concentration in the vicinity of the storm, they produce noise fields with the same vertical directionality, $f(\theta)$. As the storm sources occupy a region of limited extent on the sea surface, they fall within a window of polar angles, $w(\theta)$, and within this window, the azimuthal directionality of the noise field due to the storm sources is taken to be independent of θ and represented by the function $g(\phi)$.

Now, the full directional density function of the noise field is constructed as follows:

$$F(\theta,\phi) = [a_0 + a_1 g(\phi) w(\theta)] f(\theta), \tag{2}$$

where a_0 and a_1 are constants representing, respectively, the relative contributions from the wave breaking and storm sources. In effect, Eq. (2) states that the vertical and horizontal directivities of the storm noise are separable, given by the product of the azimuthal and vertical distributions.

In order to model the horizontal directionality of the noise generated by the storm sources, a function is required that is tractable and exhibits a maximum within the azimuthal angular interval [0, 2π]. One such function is the von Mises circular distribution[20] from directional statistics, in terms of which the horizontal directionality may be expressed as

$$g(\phi) = e^{u_1 \cos(\phi - \phi_1)}, \tag{3}$$

which shows a single peak at the azimuthal angle ϕ_1, representing the bearing to the storm center (i.e., point of maximum intensity). The real parameter u_1 governs the height of the peak, which varies as e^{u_1}, and also its width, which decreases as u_1 increases.

The vertical directionality of the noise from the surface sources is

$$f(\theta) = \begin{cases} \cos\theta & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 < \theta \le \pi, \end{cases} \tag{4}$$

which is Cron and Sherman's[17,18] directional density function for a downward- traveling, deep-water noise field. Finally, the storm window-function in Eq. (2) is chosen for simplicity to be

$$w(\theta) = \sin\left(\frac{\pi\theta}{2\theta_1}\right) \qquad [\pi/4 \le \theta_1 \le \pi/2], \tag{5}$$

where θ_1 is the polar-angle coordinate of the storm center, and the first inequality ensures that the argument of the sine function always lies in the interval [0, π]. Clearly, narrower windowing could be achieved by raising the sine function in Eq. (5) to some power $m > 1$, and this could be accommodated, at the cost of a little more mathematical complexity, by using the same analytical technique developed below for $m = 1$.

With the aid of Eqs. (3)–(5), the directional density function in Eq. (2) for the two-component, downward-traveling noise field becomes

$$F(\theta,\phi) = a_0\left\{1 + \alpha_1 e^{-u_1} e^{u_1\cos(\phi-\phi_1)} \sin\left(\frac{\pi\theta}{2\theta_1}\right)\right\}\cos\theta, \quad 0 \le \theta \le \pi/2, \tag{6}$$

where

$$\alpha_1 = \frac{a_1 e^{u_1}}{a_0} \tag{7}$$

is the ratio of the von Mises peak height to the azimuthally isotropic level from wave breaking. At the storm center, where $\theta = \theta_1$, the window function in Eq. (6) is unity and the term in parentheses reduces to

$$s(\phi) = \left\{1 + \alpha_1 e^{-u_1} e^{u_1\cos(\phi-\phi_1)}\right\}. \tag{8}$$

To illustrate the shape of the von Mises peak, this expression is plotted in Fig. 2 for various values of the parameters α_1 and u_1. The different values of ϕ_1 in Fig. 2 simply provide separation of the peaks. (Note: With α_1 and u_1 fixed, the shape of the peak is the same for all ϕ_1.) Incidentally, a function similar to a von Mises distribution but extending over polar angles in the interval $[0, \pi/2]$, was used by Liggett and Jacobson[22] to model the vertical directivity of deep-water ambient noise. In their case, they set ϕ_1 to zero, which gave rise to a fixed maximum in the direction of the upward vertical.

One of the attractive features of the von Mises distribution is that, when integrated over the interval $[0, 2\pi]$, the result is independent of ϕ_1, the position of the peak:

$$\int_0^{2\pi} e^{u_1\cos(\phi-\phi_1)} d\phi = 2\pi I_0(u_1), \tag{9}$$

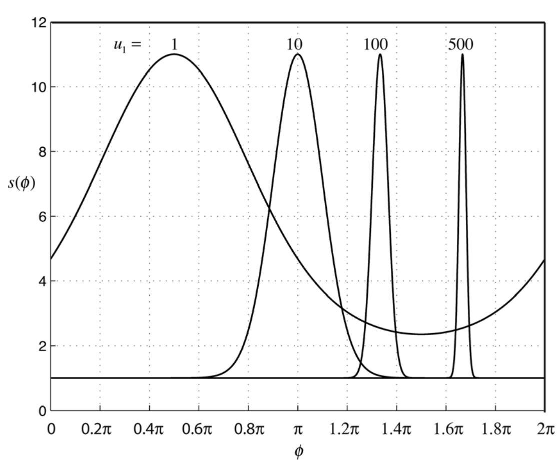

FIG. 2. The function $s(\phi)$ from Eq. (8) plotted against azimuthal angle ϕ for $\alpha_1 = 10$. From left to right: $\phi_1 = \pi/2$, $u_1 = 1$; $\phi_1 = \pi$, $u_1 = 10$; $\phi_1 = 4\pi/3$, $u_1 = 100$; $\phi_1 = 5\pi/3$, $u_1 = 500$. (Different values of ϕ_1 have been used to separate the peaks; the shape of each peak is independent of ϕ_1.)

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero. With the aid of this expression and the normalization condition in Eq. (1), the unknown scaling constant, a_0, in Eq. (6) is found to be

$$a_0 = \frac{4}{[1 + \alpha_1 e^{-u_1} I_0(u_1) A]}, \quad A = \frac{\sin\left(\frac{\pi^2}{4\theta 1}\right)}{2\left(1 - \frac{\pi^2}{16\theta_1^2}\right)}. \tag{10}$$

The directional density function in Eq. (6) is now fully characterized and may be written as

$$F(\theta,\phi) = \frac{4\left\{1 + \alpha_1 e^{-u_1} e^{u_1\cos(\phi-\phi_1)} \sin\left(\frac{\pi\theta}{2\theta_1}\right)\right\}\cos\theta}{\{1 + \alpha_1 e^{-u_1} I_0(u_1) A\}}, \quad 0 \le \theta \le \pi/2, \tag{11}$$

and for angles below the horizontal, $F(\theta,\phi)$ is zero. Equation (11) is the basis for the following derivations of the horizontal coherence function and cross-correlation function of the noise.

III. THE HORIZONTAL COHERENCE FUNCTION

For a spatially homogeneous noise field, the spatial coherence function is defined as the cross-spectral density of the fluctuations at two points in the noise field, normalized by the power spectrum, which is the same everywhere in the field. Cox[21] has derived an integral expression for the coherence function of plane-wave noise, which is valid for an arbitrary orientation of the two observation points.

For the case of interest here, in which the azimuthal coordinate is measured from the line connecting two horizontally aligned sensors, Cox's general expression for the coherence function[21] reduces to

$$\Gamma_{12}(\bar{\omega}) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} F(\theta,\phi)\exp^{-i\bar{\omega}\sin\theta\cos\phi}\sin\theta d\theta d\phi, \tag{12}$$

where $i = \sqrt{-1}$ and

$$\bar{\omega} = \frac{\omega d}{c} = 2\pi\frac{d}{\lambda} \tag{13}$$

is a normalized angular frequency. In Eq. (13), c is the speed of sound in the medium, d the distance between the sensors, and λ the acoustic wavelength at angular frequency ω. From Eq. (13), it is clear that $\bar{\omega}/2\pi$ is the aperture between the two sensors, as measured in wavelengths. Since $F(\theta,\phi)$ is taken here to be independent of frequency, the coherence function in Eq. (12) depends only on $\bar{\omega}$ rather than ω and d separately, implying that a plot of Γ_{12} versus $\bar{\omega}$ always returns the same curve, regardless of the separation of the sensors. When the sensors are coincident, the fluctuations are fully coherent, in which case the expression for the coherence function in Eq. (12) is expected to reduce to unity, a condition that does indeed follow from the normalization in Eq. (1).

On substituting Eq. (11) into Eq. (12), the coherence function takes the form

$$\Gamma_{12}(\varpi) = \frac{a_0}{4\pi}\int_0^{2\pi}\left\{1 + \alpha_1 e^{-u_1} e^{u_1\cos(\phi-\phi_1)}\sin\left(\frac{\pi\theta}{2\theta_1}\right)\right\}d\phi \times \int_0^{\pi/2} e^{-i\varpi\sin\theta\cos\phi}\sin\theta\cos\theta d\theta, \tag{14}$$

where the upper limit of $\pi/2$ on the second integral is consistent with the absence of upward-traveling noise. Several of the integrals in Eq. (14) can be expressed explicitly. First, the integral over θ is

$$I_1 = \int_0^{\pi/2} e^{-i\varpi\sin\theta\cos\phi}\sin\theta\cos\theta d\theta = i\frac{e^{-i\varpi\cos\phi}}{\varpi\cos\phi} + \frac{\left\{e^{-i\varpi\cos\phi} - 1\right\}}{\varpi^2\cos^2\phi}, \tag{15}$$

where a substitution ($y = \sin\theta$) has been made, followed by an integration by parts. This integral will be used later in the Appendix. One of the integrals over ϕ in Eq. (14) is

$$I_2 = \int_0^{2\pi} e^{-i\varpi\sin\theta\cos\phi}d\phi = 2\int_0^{\pi}\cos(\varpi\sin\theta\cos\phi)d\phi = 2\pi J_0(\varpi\sin\theta), \tag{16}$$

where the substitution $z = \cos\phi$ has yielded an integral that equates to the Bessel function, $J_0(\cdot)$, of the first kind of order zero,[23] as shown on the right-hand side. Similarly, from Eq. (14)

$$I_3 = \int_0^{\pi/2} I_2\sin\theta\cos\theta d\theta = 2\pi\int_0^{\pi/2} J_0(\varpi\sin\theta)\sin\theta\cos\theta d\theta = 2\pi\frac{J_1(\varpi)}{\varpi}, \tag{17}$$

where the substitution, $y = \sin\theta$ has produced a known integral,[24] which yields the expression on the right involving $J_1(\cdot)$, the Bessel function of the first kind of order unity. Finally, the integral

$$I_4 = \int_0^{2\pi} e^{u_1\cos(\phi-\phi_1) - i\varpi\sin\theta\cos\phi}d\phi \tag{18a}$$

reduces, after some straightforward algebraic manipulation, to the von Mises form in Eq. (9), ultimately leading to

$$I_4 = 2\pi I_0\left(\sqrt{\left[u_1\cos\phi_1 - i\varpi\sin\theta\right]^2 + u_1^2\sin^2\phi_1}\right), \tag{18b}$$

where the imaginary part of the radical in the argument of the Bessel function is less than or equal to zero.

Returning to the expression in Eq. (14), the coherence function can now be written in terms of the integrals I_3 and I_4 as follows:

$$\Gamma_{12}(\varpi) = \frac{a_0}{4\pi}\left\{I_3 + \alpha_1 e^{-u_1}\int_0^{\pi/2} I_4\cos\theta\sin\left(\frac{\pi\theta}{2\theta_1}\right)d\theta\right\} = \frac{a_0}{2}\left\{\frac{J_1(\varpi)}{\varpi} + \alpha_1 e^{-u_1}\int_0^{\pi/2} I_0\left(\sqrt{\left[u_1\cos\phi_1 - i\varpi\sin\theta\right]^2 + u_1^2\sin^2\phi_1}\right) \times \cos\theta\sin\theta\sin\left(\frac{\pi\theta}{2\theta_1}\right)d\theta\right\}. \tag{19}$$

The integral in Eq. (19) cannot be evaluated explicitly but, since the limits are finite, implying no convergence issues, it can easily be computed numerically using the trapezoidal rule or Simpson's algorithm. Figures 3 and 4 show the coherence function in Eq. (19) for various values of the parameters governing the angular position and the width of the von Mises peak in the noise field.

IV. THE CROSS-CORRELATION FUNCTION

The cross-correlation function is related to the cross-spectral density by a Fourier inversion integral.[25] Since the cross-spectral density of a spatially homogeneous noise field is the coherence function multiplied by the power spectrum, $S_0(\omega)$, the cross-correlation function may be written in the form

$$\psi_{12}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)\Gamma_{12}(\varpi)e^{i\omega\tau}d\omega, \tag{20}$$

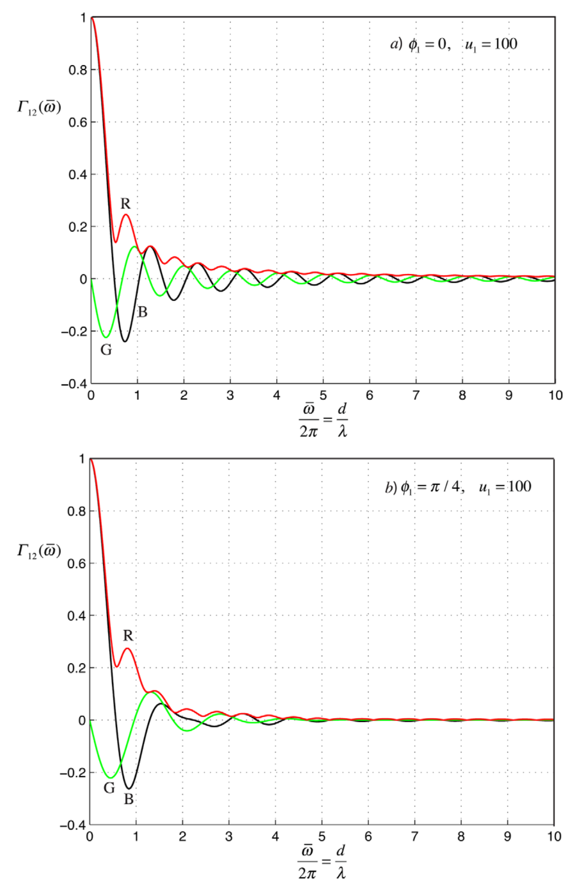

FIG. 3. (Color online) The real (curve B), imaginary (curve G), and absolute (curve R) parts of the coherence function evaluated from Eq. (19) for $\alpha_1 = 10$, $u_1 = 100$, $\theta_1 = \pi/3$ and (a) $\phi_1 = 0$; (b) $\phi_1 = \pi/4$.

where τ is the correlation delay time. Ambient noise in the ocean does not usually have a white spectrum but, assuming that a pre-whitening filter has been applied, the power spectral density of the fluctuations may be treated as uniform over an indefinitely high bandwidth, allowing the cross-correlation function to be expressed as

$$\psi_{12}(\tau) = \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \Gamma_{12}(\bar{\omega}) e^{\imath\omega\tau} d\omega, \qquad (21)$$

where S_0 is the frequency-independent spectrum of the pre-whitened noise. Although an idealization, the white noise case, as represented by Eq. (21), provides a useful baseline against which any modification of the cross correlation that may be encountered in practice, due, for example, to a restricted noise bandwidth, may be assessed.

After substituting the general expression for the coherence function in Eq. (12) into Eq. (21), the cross-correlation function takes the form

$$\begin{aligned}\psi_{12}(\tau) &= \frac{S_0}{8\pi^2}\int_0^{2\pi}\int_0^{\pi/2} F(\theta,\phi)\sin\theta d\theta d\phi \\ &\quad\times \int_{-\infty}^{\infty} e^{i\omega(\tau-\tau_d\sin\theta\cos\phi)} d\omega \\ &= \frac{\psi_0}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} F(\theta,\phi)\delta\left(\cos\phi - \frac{\tau}{\tau_d\sin\theta}\right) d\theta d\phi,\end{aligned} \qquad (22a)$$

where $\delta(\cdot)$ is the Dirac delta function and it is convenient to introduce

$$\psi_0 = \frac{S_0}{2\tau_d}, \qquad \tau_d = \frac{d}{c}. \qquad (22b)$$

The significance of ψ_o is that it is the height of the cross-correlation boxcar of isotropic white noise,[19] whilst τ_d is the acoustic travel (or retarded) time between the two sensors. In passing it is worth noting that the area under the cross-correlation function in Eq. (22a) is a constant, independent of the directionality of the noise field:

$$\int_{-\infty}^{\infty} \psi_{12}(\tau) d\tau = S_0. \qquad (23)$$

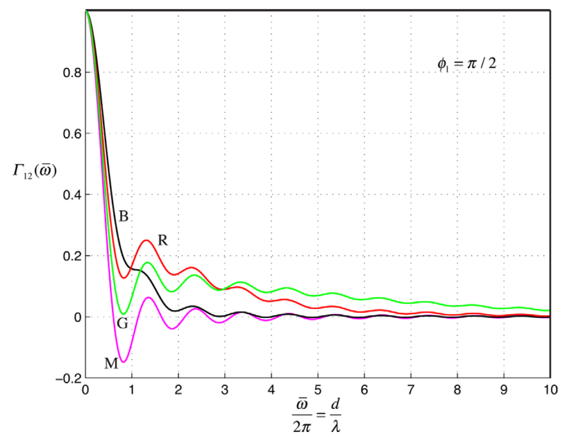

FIG. 4. (Color online) The (real) coherence function evaluated for the broadside case ($\phi_1 = \pi/2$) from Eq. (19) for $\alpha_1 = 10$, $\theta_1 = \pi/3$, and $u_1 = 1$ (curve M), $u_1 = 10$ (curve B), $u_1 = 100$ (curve R), and $u_1 = 500$ (curve G).

This result derives from the delta function in Eq. (22a) in conjunction with the normalization condition in Eq. (1).

Returning to the evaluation of the integral in Eq. (22a), when $|\tau| > \tau_d$, the delta function in the integrand lies outside the limits of the integrals and, therefore, the cross-correlation function is zero. Otherwise, the argument of the delta function, regarded as a function of ϕ, has two roots, $\phi = \gamma$ and $\phi = 2\pi - \gamma$, where

$$\gamma = \cos^{-1}\left(\frac{\tau}{\tau_d\sin\theta}\right). \qquad (24)$$

These roots contribute to the cross-correlation function only for polar angles such that $\sin\theta > |\tau|/\tau_d$. Accordingly, from Eq. (22a),

$$\psi_{12}(\tau) = \frac{\psi_0}{2\pi}\int_{\sin^{-1}(|\tau|/\tau_d)}^{\pi/2} \left\{\frac{F(\theta,\gamma) + F(\theta, 2\pi-\gamma)}{|\sin\gamma|}\right\} d\theta. \qquad (25)$$

Making a change of integration variable from θ to γ and substituting for the directional density function from Eq. (11) yields the final expression for the cross-correlation function:

$$\begin{aligned}\psi_{12}(\tau) &= \frac{4(\tau/\tau_d)\psi_o}{\pi[1+\alpha_1 e^{-u_1} I_0(u_1)A]}[u(\tau/\tau_d) - u(\tau/\tau_d)] \\ &\quad\times \int_{\cos^{-1}(\mathrm{sgn}\,\tau)}^{\cos^{-1}(\tau/\tau_d)} \frac{1+\alpha_1 e^{u_1(\cos\gamma\cos\phi_1 - 1)}\cosh(u_1\sin\gamma\sin\phi_1)\sin\left[\frac{\pi}{2\theta_1}\sin^{-1}\left(\frac{\tau}{\tau_d\cos\gamma}\right)\right]}{\cos^2\gamma} d\gamma,\end{aligned} \qquad (26)$$

where sgn$(\cdot)$ is the signum function and $u(\cdot)$ is the Heaviside unit step function. The integral in Eq. (26) can be computed using standard numerical integration techniques.

As a check on the formulation in Eq. (26), α_1 may be set to zero, a condition that removes the von Mises peak in the horizontal. The noise field is then uniform in azimuth but with a vertical directionality of the deep-water form in Eq. (4). In this situation, Eq. (26) reduces to

$$\psi_{12}(\tau) = \frac{4\psi_o}{\pi}[u(\tau+\tau_d) - u(\tau-\tau_d)]\cos[\sin^{-1}(\tau-\tau_d)], \qquad (27)$$

which, as required, is the cross-correlation function[19] for Cron and Sherman's azimuthally uniform noise field[17,18] generated by a uniform distribution of surface dipoles.

Examples of the cross-correlation function in Eq. (26) are shown in Fig. 5 for various values of the von Mises parameters. It is difficult to put a detailed physical interpretation on these curves because there is not a one-to-one mapping between the cross-correlation function and the directional density function. Instead, $\psi_{12}(\tau)$ is the result of integrating the directional density function over a range of polar angles, as exemplified by Eq. (25). This differs from the case of vertically aligned sensors, where the vertical directional density function maps directly into the cross-correlation function.[19] It is, however, clear in Fig. 5 that a pronounced maximum appears in the cross-correlation function that is associated with the von Mises peak in the noise field. The magnitude and position of the maximum depend on the width and bearing of the peak, but the area under all the curves in Fig. 5 is the same, as given by Eq. (23).

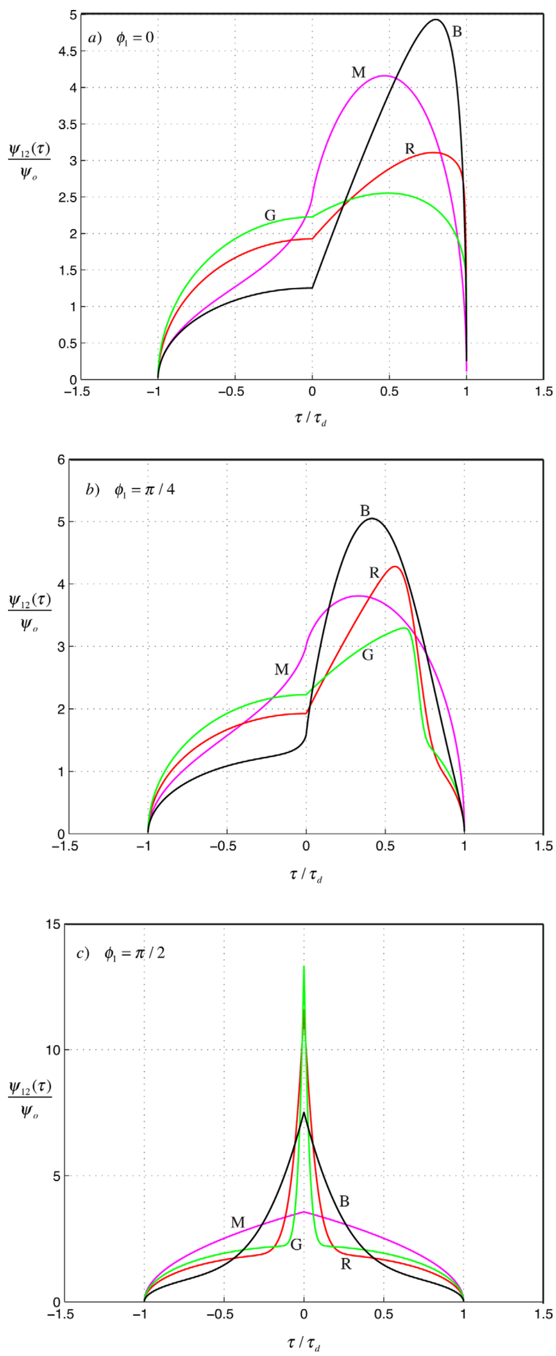

FIG. 5. (Color online) The cross-correlation function evaluated from Eq. (26) for $\alpha_1 = 10$, $\theta_1 = \pi/3$, and $u_1 = 1$ (curve M), $u_1 = 10$ (curve B), $u_1 = 100$ (curve R), $u_1 = 500$ (curve G). (a) $\phi_1 = 0$, (b) $\phi_1 = \pi/4$, and (c) $\phi_1 = \pi/2$. The area under all the curves is the same.

One of the interesting features of the curves in Fig. 5 (and indeed their derivatives with respect to τ) is the absence of delta functions that could be construed as resembling the Green's function for deterministic propagation between the two sensors. However, the discontinuous cut-offs at $\tau = \pm \tau_d$ should allow the acoustic travel time between sensors, τ_d, to be determined from either $\psi_{12}(\tau)$ or its derivative with respect to τ, in which case the sound speed of the medium could be recovered from the noise.

This conclusion is not inconsistent with several reports of travel-time determination from measurements of the cross-correlation function of ambient noise in the ocean,[26–28] although in all the reported cases the experimental conditions differed significantly from those assumed in the analysis of deep-water noise presented above. Perhaps the main departure from the assumptions of the theoretical model is that the noise in the experiments was strongly band limited. With a restricted frequency bandwidth, the cross-correlation function of the noise cannot be expected to resemble the broadband curves in Fig. 5, even if the directionality were the same.

But the noise directionality in the experiments was not the same as that in the deep-water model. For instance, the (250–750 Hz) noise in the experiments of Fried *et al.*[26] was produced by a volumetric distribution of croaker fish; Brooks and Gerstoft[27] used acoustic arrays located in the shallow water of the New Jersey Shelf to record band-limited (20–100 Hz), upward- and downward-traveling noise from a tropical storm; and Godin *et al.*[28] used pairs of hydrophones selected from adjacent vertical arrays to recover the sound speed profile in the North Pacific from band-limited noise (6–130 Hz), indicating that the noise field was not spatially homogeneous but exhibited a directionality that depended on depth.

It is interesting that, according to the deep-water model presented here, the travel time can be recovered from the cross-correlation function or its derivative, even though neither exhibits a delta function that could be identified with the Green's function. Presumably the cross-correlation functions of the band-limited, directional noise used in the experiments[26–28] are also lacking in delta functions, since there are no high-frequency Fourier components available to produce such a feature. Band limiting, however, may give rise to very sharp, oscillatory "pulses" in the cross-correlation function at delay times numerically equal to the travel time between the sensors. Such pulses, which are exemplified in Fig. 12 of Ref. 27, where they are referred to as empirical Green's functions, provide the experimental basis for recovering the sound speed in the medium from the noise.

V. MULTIPLE PEAKS IN THE HORIZONTAL

Although the von Mises distribution[20] exhibits only one peak in the angular interval [0, 2π], a superposition of several such distributions can be used to construct a directional density function with multiple peaks in azimuth. By simply summing the von Mises contributions representing each storm system, the directional density function in Eq. (6) is generalized to

$$F(\theta, \phi) = a_0 \left\{ 1 + \sum_{n=1}^{N} \alpha_n e^{-u_n} e^{u_n \cos(\phi - \phi_n)} \sin\left(\frac{\pi\theta}{2\theta_n}\right) \right\} \cos\theta, \tag{28}$$

where the subscript n identifies the associated variable with the nth von Mises distribution out of a total of N such distributions. Thus, in the nth term under the summation in Eq. (28), α_n is the ratio of the peak height to the isotropic level, u_n is a measure of the width of the peak, ϕ_n is the azimuthal direction of the peak, and θ_n is the polar angle coordinate of the storm center. The parameter a_0 in Eq. (28), representing the horizontally isotropic component of the noise field, is determined by following the same normalizing procedure as before.

Once the directional density function is known, the coherence function and cross-correlation function for the multi-peaked noise field may be derived following the same procedure as described earlier for $N = 1$. It should be clear that each von Mises distribution simply adds an associated component into the coherence function and the cross-correlation function. No new integrals are involved, beyond those already discussed for the case of a single von Mises distribution.

VI. CONCLUDING REMARKS

Ambient noise in the ocean is not usually isotropic. The directional density function, representing the spectral noise power per unit solid angle at a point in the noise field, often exhibits one or more peaks in the horizontal, associated with a variety of acoustic sources including surface shipping, localized storm systems, and underwater seismic events. In the vertical, the noise also shows some directionality, typically due to a distribution of surface dipoles associated with bubbles created by breaking waves. The result of these various noise-generating mechanisms is a three-dimensional noise field exhibiting both vertical and horizontal directivity.

The spatial coherence function and the cross-correlation function of the fluctuations are related to the directional density function through inversion integrals, and are therefore significantly influenced by the directionality of the noise. Since the coherence function and the cross-correlation function are central to various signal processing applications, it is of some interest to establish how they are affected by the vertical and horizontal directionality of the noise field.

In this article, an analysis is presented of plane-wave noise fields showing both vertical and horizontal anisotropy. Such three-dimensional noise fields are spatially homogeneous, that is, their second-order statistical measures, such as the spatial coherence and the cross-correlation function, are independent of the position in the field at which they are observed. The directional density function is treated as separable in that the overall directionality is taken to be the product of the vertical and horizontal directionalities. In the horizontal, the directionality is represented by a von Mises circular distribution taken from directional statistics; and in the vertical, the directionality is that of a simple surface-dipole model. The von Mises distribution, which exhibits a single peak, whose height, width, and azimuth are each controlled by just one parameter, has the advantage of being mathematically tractable.

On the basis of the separation between horizontal and vertical directionality, expressions are derived for the coherence function and cross-correlation function of the noise at two sensors aligned horizontally in the field. To obtain the cross-correlation function, it is assumed that the noise has been pre-whitened, that is, the power spectral density is taken to be uniform over an indefinitely large bandwidth. The expressions for the coherence function and the cross-correlation function each involve a single integral, with finite upper and lower limits, that is easy to compute using the trapezoidal rule or Simpson's algorithm.

It is interesting to note that the three-dimensional noise fields examined in this article differ from isotropic white noise in that the cross-correlation function does not exhibit delta functions that could be identified with the deterministic Green's function for acoustic propagation between the two sensors. Neither are such delta functions present in the derivative of the cross-correlation function with respect to the delay time. However, the cross-correlation function does drop abruptly to zero at positive and negative correlation delay times numerically equal to the time for an acoustic pulse to travel between the sensors. The implication here is that the travel time could be recovered from the cross-correlation function of the three-dimensional noise, from which the sound speed in the medium could be determined. This conclusion is consistent with several experiments that have been reported in the literature[26–28] in which the acoustic travel time between sensors, and hence the sound speed in the medium, has been determined from the cross-correlation function of directional ambient noise.

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APPENDIX: AN INTEGRAL REPRESENTATION OF A BESSEL FUNCTION

Suppose that the window function, $w(\theta)$, is set to unity instead of the sine function in Eq. (5). On comparing equivalent terms in the formulation of the coherence function, it is apparent that the following equality must hold:

$$\int_0^{2\pi} I_1 e^{u_1 \cos(\phi - \phi_n)} d\phi = \int_0^{\pi/2} I_4 \cos\theta \sin\theta d\theta. \tag{A1}$$

For the special case $u_1 = 0$, the integral on the right can be expressed explicitly, taking the form

$$2\pi \int_0^{\pi/2} I_0(-i\varpi \sin\theta)\cos\theta \sin\theta d\theta = 2\pi \int_0^{\pi/2} J_0(\varpi \sin\theta)\cos\theta \sin\theta d\theta = 2\pi \frac{J_1(\varpi)}{\varpi}, \quad \text{(A2)}$$

where the second integral is a known form that has already been given in Eq. (17).

Now, with the aid of the expression for I_1 in Eq. (15), it follows from Eq. (A1) with $u_1 = 0$, and from Eq. (A2) that

$$\int_0^{2\pi} \left[i \frac{e^{-i\varpi\cos\phi}}{\varpi \cos\phi} + \frac{\{e^{-i\varpi\cos\phi} - 1\}}{\varpi^2 \cos^2\phi} \right] d\phi = 2\pi \frac{J_1(\varpi)}{\varpi}. \quad \text{(A3)}$$

Since the right-hand side of Eq. (A3) is real, the imaginary part of the left-hand side must be zero. There are, in fact, two imaginary terms in the integrand and it is readily shown that each of them individually integrates to zero. The real part of the integral in Eq. (A3) must be equal to the expression on the right-hand side, that is,

$$\int_0^{2\pi} \left[\frac{\sin(\varpi\cos\phi)}{\varpi\cos\phi} + \frac{\{\cos(\varpi\cos\phi) - 1\}}{\varpi^2\cos^2\phi} \right] d\phi = 2\pi \frac{J_1(\varpi)}{\varpi}, \quad \text{(A4)}$$

which is a noteworthy result that has been derived simply by manipulating the inversion integrals in the expression for the coherence function in Eq. (14).

Although, in Eq. (A4), the integral of each term of the integrand cannot be expressed explicitly, the complete integrand integrates to the Bessel function expression on the right-hand side. As it happens, the integral in Eq. (A4) could have been evaluated by making the substitution $z = \cos\phi$, expanding each of the terms in the integrand in a Taylor series about $z = 0$, and integrating term by term. The result of this procedure is an infinite series that is identical to the series expansion of the Bessel function expression on the right-hand side of Eq. (A4).

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