On the two-point cross-correlation function of anisotropic, spatially homogeneous ambient noise in the ocean and its relationship to the Green's function

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(Received 24 September 2010; revised 1 March 2011; accepted 13 March 2011)

It is well established that the free-space Green's function can be recovered from the two-point cross-correlation function of a random noise field if the noise is white and isotropic. Ambient noise in the ocean rarely satisfies either of these conditions. However, a non-uniform spectrum could be pre-whitened by the application of a suitable filter but anisotropy cannot be so readily eliminated. To investigate the effects of vertical anisotropy, three azimuthally uniform, spatially homogeneous noise fields are analyzed, two of which are idealized, while the third is representative of ambient noise in the deep ocean. In each case, the coherence function, the cross-correlation function, and the derivative of the latter with respect to the correlation delay, are derived for vertical and horizontal alignments of the sensor pair. With vertical sensors, any step-function discontinuity in the directional density function is mapped into a delta function at an appropriate time delay in the derivative (with respect to time delay) of the cross-correlation function. No such mapping occurs with horizontal sensors. In this case, only horizontally traveling noise can generate delta functions in the derivative of the cross-correlation function, and these always appear at the retarded time on either side of the origin. © *2011 Acoustical Society of America*. [DOI: 10.1121/1.3573989]

PACS number(s): 43.30.Nb [MS] Pages: 3562–3576

I. INTRODUCTION

In 1993, Duvall *et al.*[1] advanced the idea that helioseismology, the study of the Sun's interior, could be facilitated by computing the temporal cross-correlation of random acoustic intensity fluctuations on the solar surface. The correlation technique yields the round-trip time of a trapped acoustic wave traveling from the solar surface to a point at depth in the interior, where it is refracted back to the surface. In effect, the cross-correlation function, a stochastic measure, returns the deterministic impulse response, and hence the sound speed stratification, of the Sun's interior.

Noise interferometry, whereby the impulse response or Green's function is recovered from the two-point cross-correlation function of the fluctuations in a diffuse radiation field, was introduced to the fields of ultrasonics and elastodynamic thermal noise by Weaver and Lobkis.[2–5] Their work triggered several theoretical investigations of the correlation technique,[6–9] with a surge of interest directed towards inhomogeneous media,[10–16] correlated noise sources[17] and attenuating media.[18,19] Various applications of noise interferometry have also been reported in the fields of seismology,[20–23] civil engineering,[24–26] aeronautical engineering,[27] and *in vivo* human-muscle elastography;[28] but perhaps the most extensive investigation has been performed in connection with ambient noise in the ocean,[29–37] which is the focus of interest in the present article.

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It is now well established that the free-space Green's function between two sensors is returned from the cross-correlation function [or rather, its derivative with respect to correlation delay time (Refs. 3 and 18)] of the noise fluctuations at the sensors, provided that (1) the power spectral density of the noise at the two observation points is independent of frequency (i.e., the noise is "white") over an unlimited bandwidth, and (2) the noise is isotropic. In the ocean, these two conditions are rarely, if ever, satisfied. Over a wide frequency band (approximately 1–100 kHz), where the noise is generated predominantly by wind-driven sources,[38,39] the noise spectrum tends to decrease with increasing frequency, f, falling off as $f^{-5/3}$. However, this difficulty of a non-uniform spectrum may be overcome by the straightforward expedient of pre-whitening the noise.[32]

It is not quite so easy to compensate for the directionality of the noise. Many ambient noise fields in the ocean are quite strongly anisotropic in both the vertical and the horizontal, and the directionality, along with the orientation of the sensors, has a significant influence on the cross-correlation function. Yet, in a locally homogeneous medium, neither the directionality of the noise nor the orientation of the sensors has any effect on the Green's function, which raises the question as to whether the Green's function can be recovered from the cross-correlation function under various degrees of noise anisotropy.

The purpose of this article is to examine the cross-correlation function of various spatially homogeneous, azimuthally uniform, plane-wave ambient noise fields, as determined from two sensors aligned either vertically or horizontally in the field. Although the noise fields in question

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are necessarily idealized, they provide some insight into the link between the anisotropy of the noise on the one hand and the characteristic features of the cross-correlation function on the other. To begin, expressions connecting the noise directionality, spatial coherence, cross-spectral density and spatial cross-correlation are developed, and then the well-understood case of isotropic noise is examined, since this provides a convenient introduction to the topic of noise correlation and its relationship to the Green's function. Subsequently, several anisotropic noise fields are considered.

II. NOISE DIRECTIONALITY, COHERENCE, AND CORRELATION

Ambient noise in the ocean is commonly represented as a superposition of independent, uncorrelated plane-waves propagating in all directions.[40] Such a plane-wave noise field is spatially homogeneous, that is, its second-order statistical measures, notably the power- and cross-spectral densities, the coherence function and the cross-correlation function, are independent of position in the field. A plane-wave noise field may be constructed theoretically by placing a random distribution of independent, point sources on a sphere of infinite radius, as illustrated in Fig. 1. At the center of the sphere, where the receiver station is located, the noise directionality may be controlled by adjusting the density of the sources as a function of angular position on the sphere. Such an idea dates back several decades, having been used by Faran and Hills[41] and Jacobson[42] to model plane-wave acoustic noise fields.

At the receiver station, the noise intensity per steradian is represented by the directional density function, $F(\theta, \phi, \omega)$, where θ is the polar angle measured from the upward vertical, ϕ is the azimuthal angle, and ω is angular frequency. It is assumed here that the noise sources are positive and negative impulses (delta functions), each having an infinitely broad, flat spectrum, and which together give rise to a zero-mean, white-noise field having a directional density function that is independent of frequency. In addition, the noise is taken to be azimuthally uniform, allowing the directional density function to be expressed as $F(\theta)$, that is, as a function

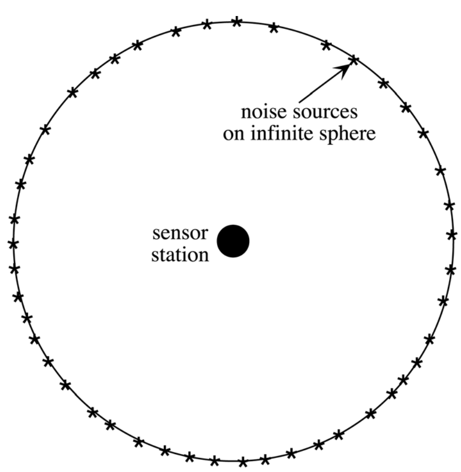

FIG. 1. Infinite-radius sphere of independent, randomly distributed point sources with the sensor station located at the center.

of just a single variable, the polar angle, θ, extending over the interval $0 \le \theta \le \pi$. The upward and downward verticals correspond to $\theta = 0$ and $\theta = \pi$, respectively, and the horizontal is at $\theta = \pi/2$. A standard normalization of the directional density function[40] is achieved by integrating $F(\theta)$ over all solid angles and equating the result to 4π, which, for the azimuthally uniform noise fields considered here, yields the condition

$$\frac{1}{2}\int_0^{\pi} F(\theta)\sin\theta d\theta = 1. \tag{1}$$

Consider two sensors, designated 1 and 2, at positions in the plane-wave noise field where the fluctuations are represented by the time series, $x_1(t)$ and $x_2(t)$, respectively. Letting the Fourier transforms of these time series be $X_1(\omega)$ and $X_2(\omega)$, then the bilateral power spectral density of the noise at the two sensors is

$$S_{jj}(\omega) = \frac{\overline{|X_j(\omega)|^2}}{T}, \tag{2a}$$

where the subscript $j = 1$ or 2, T is the observation time used to create the Fourier transforms, and the overbar denotes an ensemble average. Similarly, the cross-spectral density of the noise fluctuations at sensors 1 and 2 is

$$S_{12}(\omega) = \frac{\overline{X_1(\omega)X_2^*(\omega)}}{T}, \tag{2b}$$

where the asterisk denotes complex conjugation. The coherence function, $\Gamma_{12}(\omega)$, is defined as the cross-spectral density normalized to the geometric mean of the power spectral densities at the two sensors

$$\Gamma_{12}(\omega) = \frac{S_{12}(\omega)}{\sqrt{S_{11}(\omega)S_{22}(\omega)}} = \frac{S_{12}(\omega)}{S_0(\omega)}, \tag{3}$$

where the two power spectra under the radical have both been set equal to $S_0(\omega)$, since a plane-wave noise field is spatially homogeneous and hence the power spectral density is the same everywhere in the field.

Cox[40] has performed an elegant analysis which leads to an expression for the coherence function, $\Gamma_{12}(\omega)$, in terms of the directional density function, $F(\theta)$, valid for an arbitrary orientation of the two sensors in the plane-wave noise field. His results for the two cases of interest here, namely vertical and horizontal alignment of the sensors, are, respectively,

$$\Gamma_{12}(\omega) = \frac{1}{2}\int_0^{\pi} F(\theta)e^{-i\bar{\omega}\cos\theta}\sin\theta\, d\theta \quad \text{(vertical)} \tag{4a}$$

and

$$\Gamma_{12}(\omega) = \frac{1}{2}\int_0^{\pi} F(\theta)J_0(\bar{\omega}\sin\theta)\sin\theta\, d\theta \quad \text{(horizontal)}, \tag{4b}$$

where $i = \sqrt{-1}$ and $J_0(\cdots)$ is the Bessel function of the first kind of order zero. In both these expressions, a normalized angular frequency appears, defined as

$$\bar{\varpi} = \frac{\omega d}{c}, \quad (5)$$

where d is the separation of the sensors and c is the speed of sound in the medium. In general, the coherence function from the vertically aligned sensors is complex, with the real and imaginary parts, respectively, arising from the symmetrical and anti-symmetrical components of $F(\theta)$ about the horizontal ($\theta = \pi/2$). The coherence function in the case of the horizontally aligned sensors is always real. Notice that, in the limit as $\bar{\varpi} \to 0$, both the expressions in Eq. (4) reduce to the normalization condition in Eq. (1), as indeed they should since the normalization is independent of the orientation of the sensors.

The cross-correlation function, $\psi_{12}(\tau)$, between the fluctuations at sensors 1 and 2 is given by a Fourier transform of the cross-spectral density,[43]

$$\psi_{12}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{12}(\omega)e^{i\omega\tau}d\omega, \quad (6)$$

where τ is the correlation delay time between the time series $x_1(t)$ and $x_2(t)$. By substituting from Eq. (3), the cross-correlation function can be expressed in terms of the coherence function as follows:

$$\psi_{12}(\tau) = \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \Gamma_{12}(\omega)e^{i\omega\tau}d\omega, \quad (7)$$

where the power spectrum of the noise has been taken outside the integral on the grounds that the noise sources are assumed to be white and hence the power spectral density of the noise is independent of frequency.

By combining Eqs. (4) and (7), the cross-correlation function may be expressed in terms of the directional density function, with the following results for vertically and horizontally aligned sensors

$$\psi_{12}(\tau) = \frac{S_0}{4\pi}\int_{-\infty}^{\infty}\int_0^{\pi} F(\theta)e^{i\omega(\tau-\tau_d\cos\theta)}\sin\theta d\theta d\omega \quad \text{(vertical)} \quad (8a)$$

and

$$\psi_{12}(\tau) = \frac{S_0}{4\pi}\int_{-\infty}^{\infty}\int_0^{\pi} F(\theta)J_0(\omega\tau_d\sin\theta)e^{i\omega\tau} \times \sin\theta d\theta d\omega \quad \text{(horizontal)}, \quad (8b)$$

where

$$\tau_d = \frac{d}{c} \quad (9)$$

is the acoustic travel time (or retarded time) over the distance, d, between the sensors. To extract the Green's function, the derivative (with respect to τ) of the cross-correlation function is taken.[18] Differentiation under the integral signs in Eq. (8) yields

$$\psi_{12}'(\tau) \equiv \frac{d\psi_{12}(\tau)}{d\tau} = \frac{iS_0}{4\pi}\int_{-\infty}^{\infty}\int_0^{\pi} F(\theta)\omega e^{i\omega(\tau-\tau_d\cos\theta)} \times \sin\theta d\theta d\omega \quad \text{(vertical)} \quad (10a)$$

and

$$\psi_{12}'(\tau) \equiv \frac{d\psi_{12}(\tau)}{d\tau} = \frac{iS_0}{4\pi}\int_{-\infty}^{\infty}\int_0^{\pi} F(\theta)J_0(\omega\tau_d\sin\theta)\omega e^{i\omega\tau} \times \sin\theta d\theta d\omega \quad \text{(horizontal)}, \quad (10b)$$

where the prime denotes a derivative with respect to the correlation delay time, τ.

Once the directional density function has been specified, the integrals over the polar angle, θ, in Eq. (8) may be evaluated, followed by the integrals over frequency, to obtain the cross-correlation function. Alternatively, since $F(\theta)$ is taken to be independent of frequency, the integrals over frequency in Eq. (8) may be evaluated first to yield

$$\int_{-\infty}^{\infty} e^{i\omega(\tau-\tau_d\cos\theta)}d\omega = 2\pi\delta(\tau-\tau_d\cos\theta) \quad (11a)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} J_0(\omega\tau_d\sin\theta)e^{i\omega\tau}d\omega &= 2\int_0^{\infty} J_0(\omega\tau_d\sin\theta)\cos\omega\tau d\omega, \\ &= \begin{cases} \dfrac{2}{\sqrt{\tau_d^2\sin^2\theta-\tau^2}}, & |\tau| < \tau_d\sin\theta \\ 0, & |\tau| > \tau_d\sin\theta, \end{cases}\end{aligned} \quad (11b)$$

where $\delta(\cdots)$ is the Dirac delta function, and the integral in Eq. (11b) is a known form that can be found in tables of integrals.[44] On substituting these expressions back into Eq. (8), some straightforward algebraic manipulation returns the following expressions for the cross-correlation functions:

$$\begin{aligned}\psi_{12}(\tau) &= \frac{S_0}{2\tau_d}\int_{-1}^{1} F(\cos^{-1}y)\delta\left(y-\frac{\tau}{\tau_d}\right)dy \\ &= \frac{S_0}{2\tau_d}F\left(\cos^{-1}\left\{\frac{\tau}{\tau_d}\right\}\right) \quad \text{(vertical)}\end{aligned} \quad (12a)$$

and

$$\begin{aligned}\psi_{12}(\tau) &= \frac{S_0}{2\pi\tau_d}[u(\tau+\tau_d)-u(\tau-\tau_d)] \\ &\quad \times \int_{\theta_1}^{\pi-\theta_1} \frac{F(\theta)}{\sqrt{\cos^2\theta_1-\cos^2\theta}}\sin\theta d\theta \\ &= \frac{S_0}{2\pi\tau_d}[u(\tau+\tau_d)-u(\tau-\tau_d)] \\ &\quad \times \int_0^{\pi} F(\cos^{-1}\{\cos\theta_1\cos\phi\})d\phi, \quad \text{(horizontal)},\end{aligned} \quad (12b)$$

where

$$\theta_1 = \sin^{-1}\left\{\frac{|\tau|}{\tau_d}\right\}, \quad \cos\theta_1 = \sqrt{1-(\tau/\tau_d)^2}. \qquad (13)$$

The limits on the first integral in Eq. (12b) follow from the inequalities in Eq. (11b), which also account for the Heaviside unit step-functions representing the correlation window spanning the interval $[-\tau_d, \tau_d]$. When it comes to evaluating the cross-correlation function, given a specific functional form for $F(\theta)$, the formulation in either Eq. (8) or (12) may be used, whichever is more convenient.

The implication of the second expression in Eq. (12a) is worth noting: with vertically aligned sensors in an azimuthally uniform noise field, the cross-correlation function is a direct measure of the directional density function. Clearly, this relationship could form the basis of an inversion procedure for recovering the vertical directionality of the noise. By setting

$$\alpha = \cos^{-1}\left\{\frac{\tau}{\tau_d}\right\}, \qquad (14)$$

it is evident from Eq. (12a) that

$$F(\alpha) = \frac{2\tau_d}{S_0}\psi_{12}(\tau_d\cos\alpha), \quad 0 \le \alpha \le \pi. \qquad (15)$$

According to this expression, the cross-correlation function maps into the directional density function and vice versa. Once the cross-correlation function has been determined from a measurement of the noise fluctuations at the two vertically separated sensors, the shape of the directional density function is revealed immediately from the relationship in Eq. (15).

With horizontally aligned sensors the situation is a little more complex, as can be seen in Eq. (12b), where the cross-correlation function is expressed as an integral of the directional density function taken over finite limits. Nevertheless, even in this case, the relationship between the cross-correlation function and the vertical directionality of the noise is readily evaluated once a functional form for $F(\theta)$ has been specified.

III. ISOTROPIC NOISE

As a check on the expressions derived above, consider the well-documented case of isotropic noise. For such a noise field, the directional density function is independent of the polar angle, extending uniformly over the finite angular interval $0 \le \theta \le \pi$, that is,

$$F(\theta) = [u(\theta) - u(\theta-\pi)]F_0 = [u(\theta) - u(\theta-\pi)], \qquad (16)$$

where the constant F_0 has been set to unity, consistent with the normalization condition in Eq. (1).

Symmetry dictates that both the coherence function and the cross-correlation function be independent of the orientation of the sensors. This condition is indeed satisfied by Eqs. (4a) and (4b), since both return the familiar expression,

$$\Gamma_{12} = \frac{\sin\bar{\omega}}{\bar{\omega}}. \qquad (17a)$$

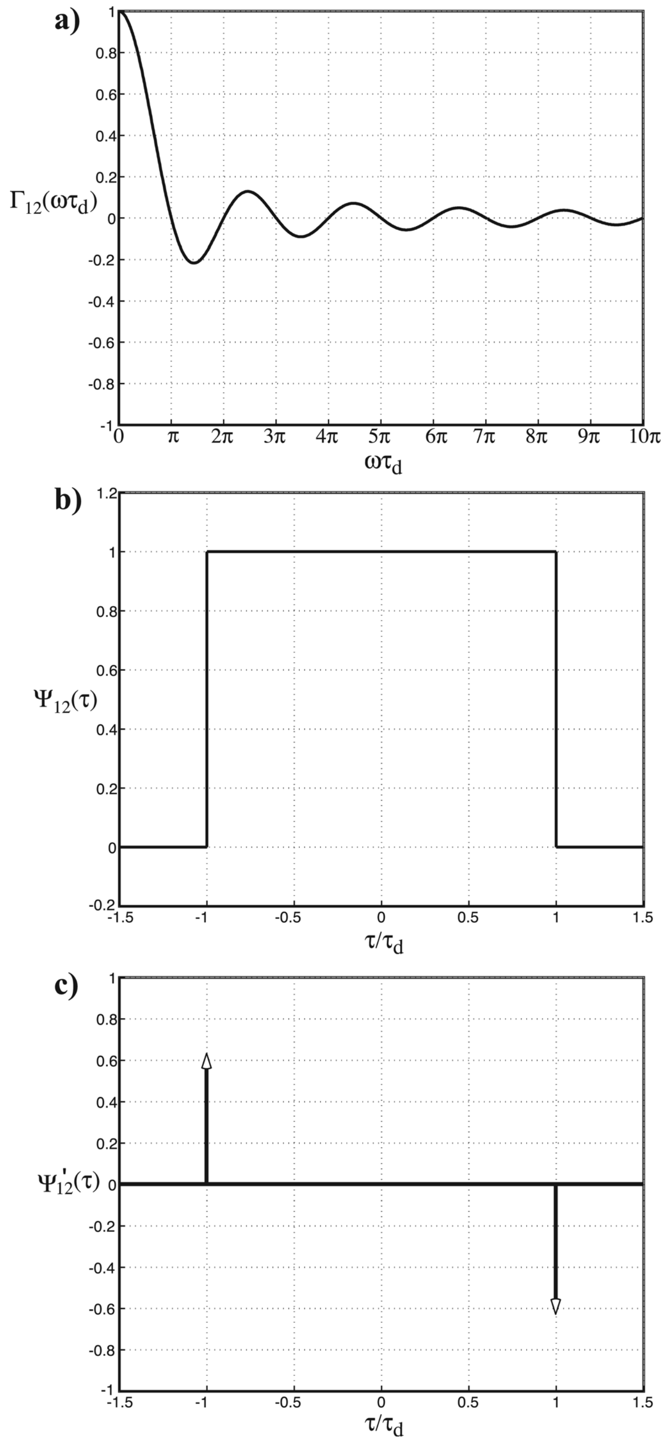

FIG. 2. Isotropic noise: (a) coherence function evaluated from Eq. (17a); (b) normalized cross-correlation function from Eq. (17b); (c) derivative of the normalized cross-correlation function with respect to delay time from Eq. (17c) (open arrows represent delta functions).

As shown in Fig. 2(a), the zero-crossings occur when $\bar{\omega}$ is a multiple of π, with the first zero corresponding to half-wavelength spacing of the sensors.

Similarly, it is immediately evident that Eqs. (12a) and (12b) both return the same result for the cross-correlation function

$$\psi_{12}(\tau) = \frac{S_0}{2\tau_d}[u(\tau+\tau_d) - u(\tau-\tau_d)], \qquad (17b)$$

which is a symmetric (even) function of τ, taking the form of a window of uniform amplitude, or boxcar, extending between $\tau = \pm\tau_d$, as illustrated in Fig. 2(b). The derivative of Eq. (17b) with respect to the delay time is

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d}[\delta(\tau+\tau_d) - \delta(\tau-\tau_d)], \tag{17c}$$

which is an anti-symmetric (odd) function of τ, with delta functions of opposite sign appearing at the retarded time on either side of the origin [Fig. 2(c)]. Such behavior is of course well known,[7,16,33] reflecting the fact that, in isotropic noise, endfire noise components of equal power propagate in opposite directions along the line joining sensors 1 and 2. Incidentally, in Fig. 2 the cross-correlation function and its derivative are normalized to $S_0/2\tau_d$ and the delay time is normalized to τ_d. All subsequent plots of noise cross-correlation functions and their derivatives are normalized in the same way.

The (acoustic pressure) free-space Green's function for propagation between sensors 1 and 2 is

$$g_{12}(t) = \frac{1}{d}\delta(t-\tau_d), \tag{18a}$$

where t is time and a scaling constant on the right has been set to unity. On comparing Eqs. (17c) and (18a), it is evident that, to within a scaling constant,

$$\psi'_{12}(\tau) = g_{12}(-t) - g_{12}(t), \tag{18b}$$

which is the result connecting the derivative of the cross-correlation function of isotropic noise and the free-space Green's function that has been discussed by several previous authors.[18,32] Clearly, the anti-symmetric delta functions in Eq. (17c), that is, the delta functions that are identified with the free-space Green's function, derive directly from the step-function discontinuities at either end ($\tau = \pm\tau_d$) of the cross-correlation boxcar. These step-function discontinuities occur because the directional density function itself takes the form of a boxcar, as expressed in Eq. (16), extending over the finite angular interval $0 \le \theta \le \pi$ with step-function discontinuities at either end ($\theta = 0$ and $\theta = \pi$).

Ambient noise fields in the ocean are not usually isotropic but they are often spatially homogeneous or, at least, quasi-homogeneous over typical distances separating the hydrophones used in measuring the two-point cross-correlation function. Such noise fields may be represented as a random superposition of plane-waves propagating in all directions. Several simple examples of plane-wave, azimuthally uniform noise fields are discussed below, with a view to establishing the effect of vertical anisotropy on the cross-correlation function returned by a pair of sensors aligned either vertically or horizontally in the field. Weaver *et al.*[15] discussed co-axial (vertical), but not perpendicular (horizontal), receivers in an azimuthally uniform noise field. Their representation of the noise directionality as a cosine Fourier series differs from the formulations discussed below.

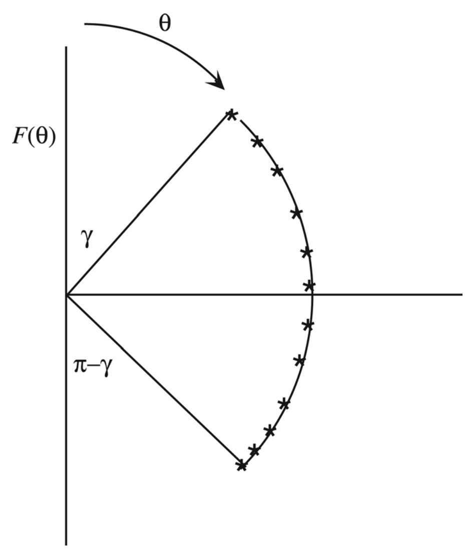

FIG. 3. Directional density function showing a horizontal noise lobe generated by distant sources distributed uniformly (statistically) over the interval $\gamma \le \theta \le (\pi - \gamma)$.

IV. NOISE LOBE IN THE HORIZONTAL

Figure 3 illustrates an azimuthally uniform, anisotropic noise field with a single lobe symmetrically placed about the horizontal and extending between the polar angles γ and $(\pi - \gamma)$. Within the lobe, the directional density function is independent of the polar angle and outside it is zero

$$F(\theta) = [u(\theta-\gamma) - u(\theta-\pi+\gamma)]F_0, \tag{19a}$$

where $0 \le \gamma \le \pi/2$ and from the normalization condition in Eq. (1),

$$F_0 = \frac{1}{\cos\gamma}. \tag{19b}$$

Note that isotropic noise is a special case of Eq. (19a) for which $\gamma = 0$.

A. Vertical sensors

From Eq. (4a), the coherence function is

$$\Gamma_{12}(\omega) = \frac{F_0}{2}\int_{\gamma}^{\pi-\gamma} e^{-i\bar{\omega}\cos\theta}\sin\theta d\theta = \frac{\sin(\bar{\omega}\cos\gamma)}{\bar{\omega}\cos\gamma}, \tag{20a}$$

where the integral has been evaluated by making the substitution $y = \cos\theta$. The coherence function in Eq. (20a) is real, consistent with the even symmetry of the noise lobe about the horizontal and, with $\gamma > 0$, its zero-crossings are less densely packed than those in the isotropic case. The zeros occur when $\bar{\omega}$ is a multiple of $\pi/\cos\gamma$, as illustrated in Fig. 4(a) for the case $\gamma = \pi/4$.

The cross-correlation function is given directly by Eq. (12a) as

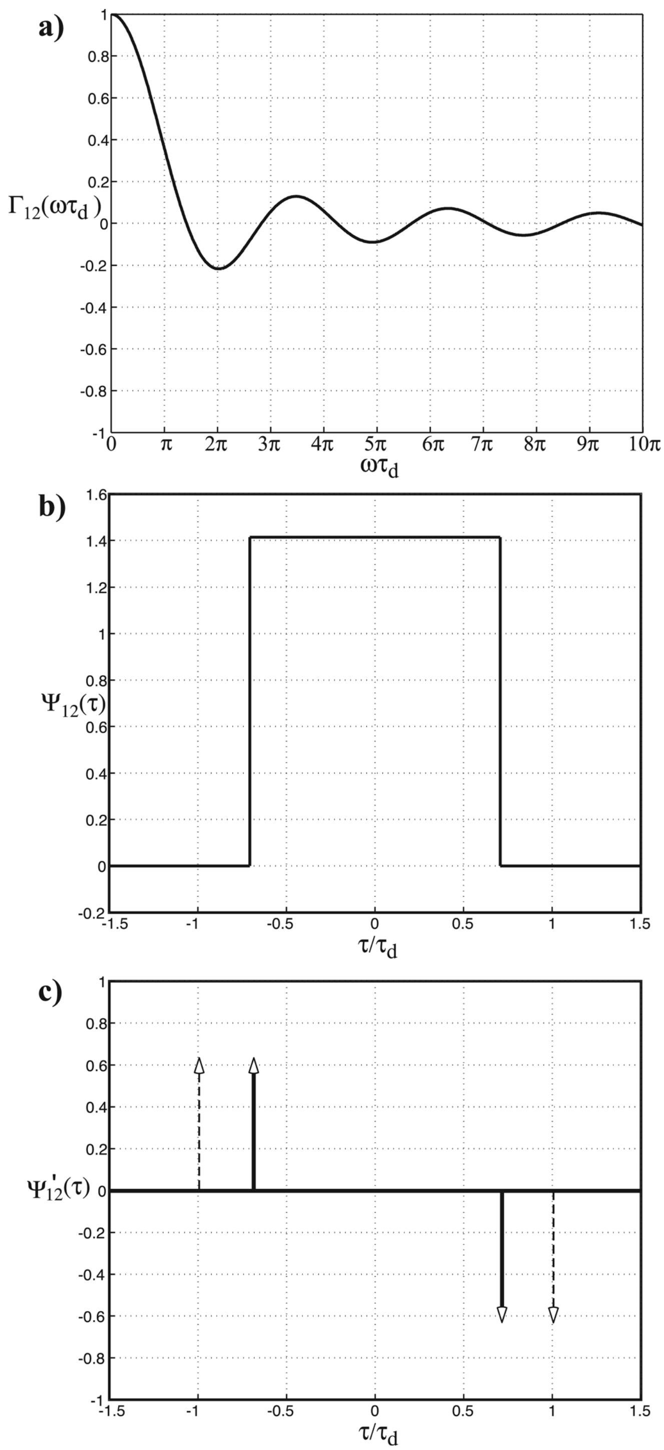

FIG. 4. Horizontal noise lobe, $\gamma = \pi/4$, and vertically aligned sensors: (a) coherence function, evaluated from Eq. (20a); (b) normalized cross-correlation function, evaluated from Eq. (20b); (c) schematic showing the delta functions in Eq. (20c) (solid open arrows) and the Greens functions, $g_{12}(-t)$ and $-g_{12}(t)$, in Eq. (18b) (dashed open arrows).

$$\psi_{12}(\tau) = \frac{S_0}{2\tau_d \cos\gamma}[u(\tau + \tau_d\cos\gamma) - u(\tau - \tau_d\cos\gamma)], \tag{20b}$$

the derivative of which is

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d \cos\gamma}[\delta(\tau + \tau_d\cos\gamma) - \delta(\tau - \tau_d\cos\gamma)]. \tag{20c}$$

The cross-correlation function in Eq. (20b) and a schematic of its derivative in Eq. (20c) are shown in Figs. 4(b) and 4(c), respectively. When $\gamma \neq 0$, it is clear that the delta functions in Eq. (20c) are offset from the free-space Green's functions $g_{12}(-t)$ and $-g_{12}(t)$ of Eq. (18b). The reason, of course, is the reduced angular width of the directional density function, which gives rise to a narrower boxcar in the cross-correlation function. Obviously, the delta functions in Eq. (20c) are associated with the step-function discontinuities at either end of the cross-correlation boxcar, which occur at delay times $\pm\tau_d \cos\gamma$. This is simply the longest time taken by any of the noise wave fronts to pass through both sensors.

Note that Eqs. (20a)–(20c) all reduce, correctly, to their isotropic counterparts when $\gamma = 0$. The maximum travel time of a wave front between the sensors is then τ_d, associated with noise rays propagating along the line joining the receivers, the cross-correlation boxcar extends over the interval $-\tau_d \leq \tau \leq \tau_d$, and the anti-symmetric delta functions in the derivative of the cross-correlation function occur at $\tau = \pm\tau_d$. As pointed out earlier, these delta functions are the ones that have been identified with the free-space Green's function by previous authors.[18,32]

B. Horizontal sensors

From Eq. (4b), the coherence function is

$$\begin{aligned}\Gamma_{12}(\omega) &= \frac{1}{2}\int_0^{\pi} J_0(\bar{\omega}\sin\theta)F(\theta)\sin\theta d\theta \\ &= \frac{F_0}{2}\int_{\gamma}^{\pi-\gamma} J_0(\bar{\omega}\sin\theta)\sin\theta d\theta \\ &= \frac{1}{\cos\gamma}\int_0^{\cos\gamma} J_0\left(\bar{\omega}\sqrt{1-y^2}\right)dy.\end{aligned} \tag{21}$$

For $\gamma > 0$, the integral here cannot be expressed explicitly, but it is readily evaluated numerically using the trapezoidal formula or Simpson's rule. The resultant coherence curve is plotted in Fig. 5(a) for the case $\gamma = \pi/4$.

The cross-correlation function may be evaluated from Eq. (12b) by making the substitution

$$\cos q = \cos\theta_1 \cos\phi, \tag{22}$$

where θ_1 is a function of the correlation delay time, as defined in Eq. (13). It follows that the cross-correlation function may be written as

$$\psi_{12}(\tau) = \frac{S_0}{2\pi\tau_d}[u(\tau + \tau_d) - u(\tau - \tau_d)] \times \int_{\theta_1}^{\pi-\theta_1} \frac{F(q)\sin q}{\sqrt{\cos^2\theta_1 - \cos^2 q}}dq. \tag{23a}$$

On taking the directional density function, Eq. (19), outside the integral, this expression becomes

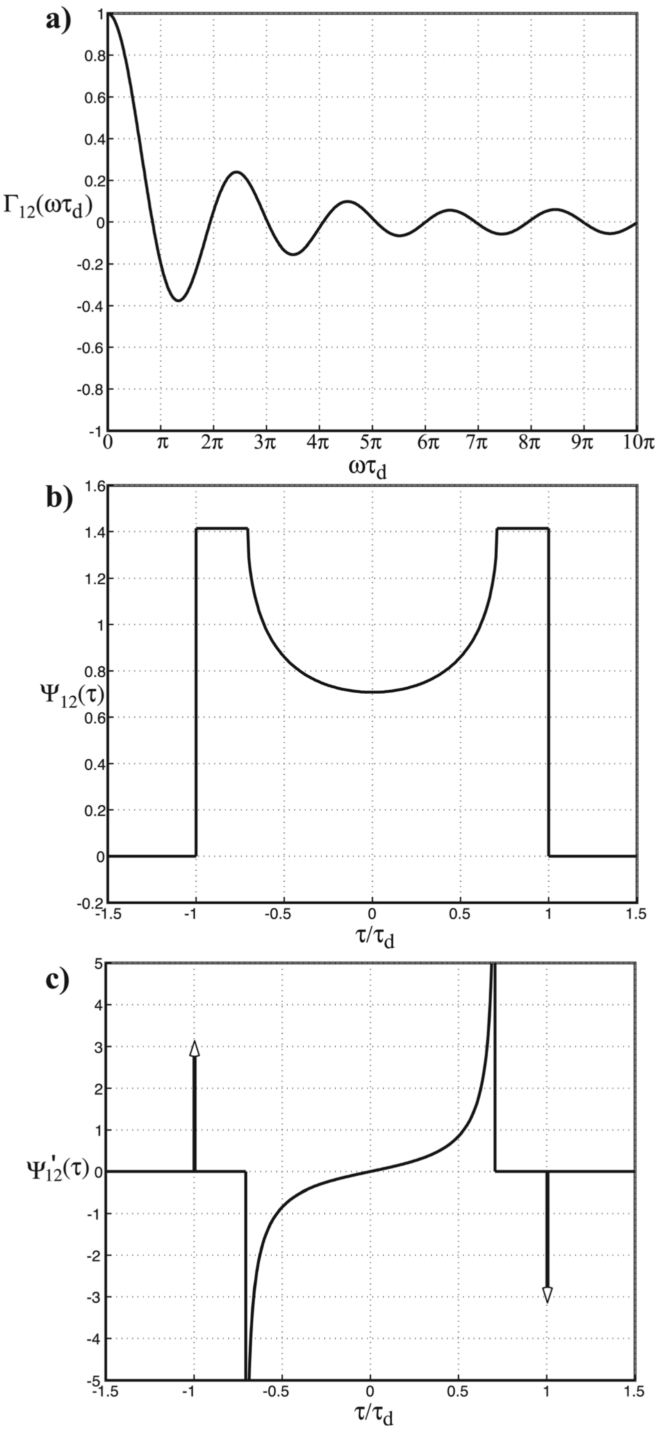

FIG. 5. Horizontal noise lobe, $\gamma = \pi/4$, and horizontally aligned sensors: (a) coherence function computed from Eq. (21); (b) normalized cross-correlation function from Eq. (24a); (c) derivative of the normalized cross-correlation function from Eq. (24b).

$$\psi_{12}(\tau) = \frac{S_0}{2\pi\tau_d \cos\gamma}[u(\tau+\tau_d) - u(\tau-\tau_d)] \times \int_{\eta}^{\pi-\eta} \frac{\sin q}{\sqrt{\cos^2\theta_1 - \cos^2 q}} dq, \tag{23b}$$

where η appearing in the limits on the integral is the greater of θ_1 and γ (i.e., the limits depend on the correlation delay time, τ). The integral itself can be evaluated explicitly by making an elementary substitution, the final result for the cross-correlation function being

$$\psi_{12}(\tau) = \frac{S_0}{2\tau_d \cos\gamma}[u(\tau+\tau_d) - u(\tau-\tau_d)] - \frac{S_0}{2\tau_d \cos\gamma}[u(\tau+\tau_d\sin\gamma) - u(\tau-\tau_d\sin\gamma)] \times \left\{1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\cos\gamma}{\sqrt{1-(\tau/\tau_d)^2}}\right)\right\}. \tag{24a}$$

When $\gamma = 0$, representing isotropic noise, the second expression on the right is zero, because the boxcar has zero width, and Eq. (24a) reduces correctly to the functional form in Eq. (17b). It is interesting to note that the double integral for the cross-correlation function in Eq. (8b) is given by the analytical function in Eq. (24a), even though the integral over angle cannot itself be expressed explicitly. The derivative of the cross-correlation function in Eq. (24a), taken with respect to the correlation delay time, τ, is

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d \cos\gamma}\left\{[\delta(\tau+\tau_d) - \delta(\tau-\tau_d)] + \frac{2}{\pi}[u(\tau+\tau_d\sin\gamma) - u(\tau-\tau_d\sin\gamma)] \times \frac{(\tau/\tau_d)\cos\gamma}{\tau_d[1-(\tau/\tau_d)^2]\sqrt{\sin^2\gamma - (\tau/\tau_d)^2}}\right\}. \tag{24b}$$

Figure 5(b) shows Eq. (24a) plotted as a function of the delay time for the case $\gamma = \pi/4$. Step discontinuities occur when $|\tau| = \tau_d \sin\gamma$, and step-function discontinuities are present at either end of the correlation window, where $|\tau| = \tau_d$. As shown in Fig. 5(c), the latter give rise to anti-symmetric delta functions in $\psi'_{12}(\tau)$ that match those in the free-space Green's function expression in Eq. (18), which is only to be expected since, in this case, noise components propagate in both directions along the line connecting the sensors. The step discontinuities at $|\tau| = \tau_d \sin\gamma$, however, are not step-functions and hence do not yield delta functions in $\psi'_{12}(\tau)$, although they do give rise to infinities associated with the radical in the denominator on the right of Eq. (24b). Clearly, if the additional structure were ignored, the delta functions in $\psi'_{12}(\tau)$ could be identified with the Green's function of an isotropic medium.

V. NOISE NOTCH IN THE HORIZONTAL

Figure 6 shows a noise field with a notch placed symmetrically about the horizontal, on either side of which the noise distribution is statistically uniform. The directional density function for this noise distribution is

$$F(\theta) = [u(\theta) - u(\theta-\beta) + u(\theta-\pi+\beta) - u(\theta-\pi)]F_1, \tag{25a}$$

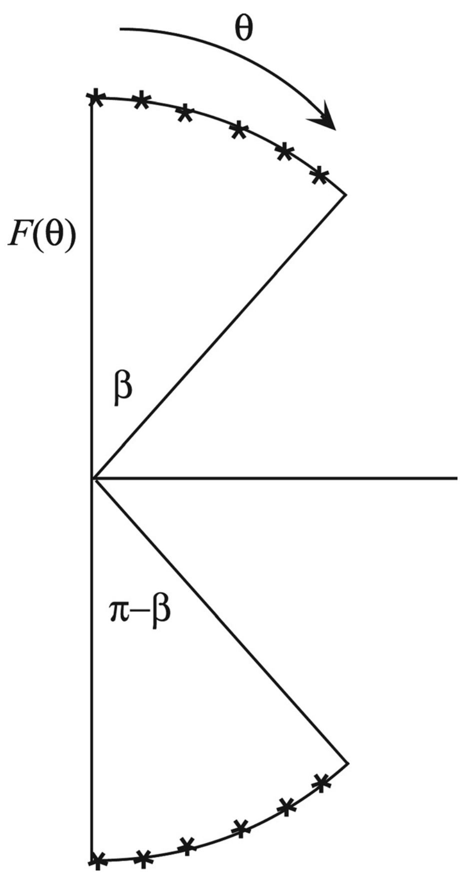

FIG. 6. Directional density function showing noise notch, symmetrical about the horizontal and extending over the interval $\beta \le \theta \le (\pi - \beta)$, with uniformly distributed noise on either side.

where $0 \le \beta \le \pi/2$ and F_1 is a constant, given by the normalization condition in Eq. (1) as

$$F_1 = \frac{1}{(1 - \cos\beta)}. \tag{25b}$$

Note that Eq. (25a) represents the special case of isotropic noise when $\beta = \pi/2$.

A. Vertical sensors

From Eq. (4a), the coherence function of the noise may be written almost immediately as

$$\Gamma_{12}(\omega) = \frac{2\sin[\bar{\omega}(1 - \cos\beta)/2]\cos[\bar{\omega}(1 + \cos\beta)/2]}{\bar{\omega}(1 - \cos\beta)}, \tag{26a}$$

which reduces correctly to the isotropic form in Eq. (17a) when $\beta = \pi/2$. Since the noise is symmetrical about the horizontal, the coherence function in Eq. (26a) is real, taking the form shown in Fig. 7(a) for the case $\beta = \pi/4$.

The cross-correlation function is readily established, either from Eqs. (8a) or (12a)

$$\begin{aligned}\psi_{12}(\tau) &= \frac{S_0}{4\pi[1 - \cos\beta]} \\ &\quad \times \int_{-\infty}^{\infty} \frac{[e^{i\bar{\omega}} - e^{-i\bar{\omega}} + e^{-i\bar{\omega}\cos\beta} - e^{i\bar{\omega}\cos\beta}]}{i\bar{\omega}} e^{i\omega\tau} d\omega \\ &= \frac{S_0}{2\tau_d[1 - \cos\beta]}[u(\tau + \tau_d) - u(\tau + \tau_d\cos\beta) \\ &\quad + u(\tau - \tau_d\cos\beta) - u(\tau - \tau_d)].\end{aligned} \tag{26b}$$

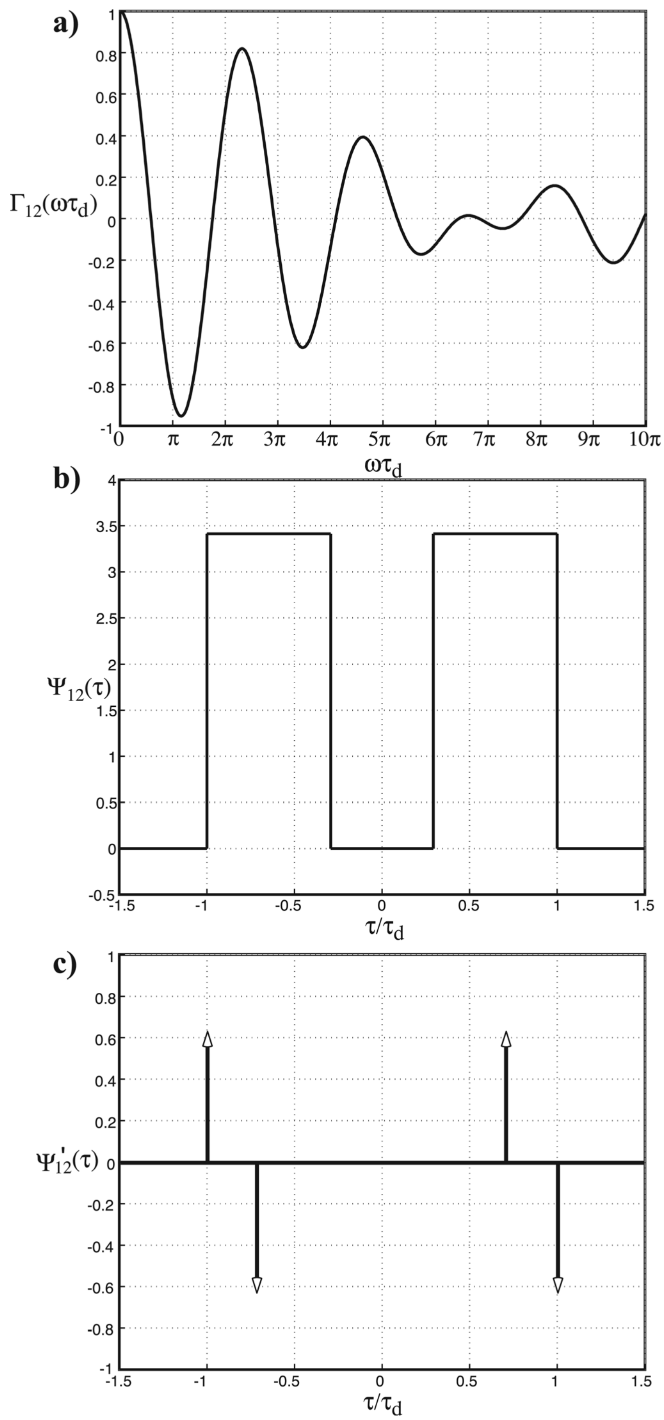

FIG. 7. Horizontal noise notch, $\beta = \pi/4$, and vertically aligned sensors: (a) coherence function evaluated from Eq. (26a); (b) normalized cross-correlation function from Eq. (26b); (c) derivative with respect to time delay of the normalized cross-correlation function, from Eq. (26c).

This function is plotted versus the correlation delay time in Fig. 7(b). It consists of two boxcars, each associated with one of the noise lobes shown in Fig. 6. The derivative of the cross-correlation function with respect to τ is immediately found to be

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d[1-\cos\beta]}[\delta(\tau+\tau_d) - \delta(\tau+\tau_d\cos\beta) + \delta(\tau-\tau_d\cos\beta) - \delta(\tau-\tau_d)], \tag{26c}$$

and this expression is plotted in Fig. 7(c). The delta functions appearing at either end of the correlation window in Eq. (26c) match those in the free-space Green's function expression in Eq. (18); but the two additional anti-symmetric delta functions appearing at $\tau = \pm\tau_d \cos\beta$ do not conform to the Green's function. These additional delta functions are associated with the step-function discontinuities in the noise distribution at $\theta = \beta$ and $\theta = \pi - \beta$. For the case $\beta = \pi/2$, the two boxcars in Fig. 7(b) merge into one, extending over the interval $-\tau_d \le \tau \le \tau_d$, and the inner delta functions in Fig. 7(c) cancel each other, leaving only the delta functions at $\tau = \pm\tau_d$, as expected for isotropic noise.

B. Horizontal sensors

The coherence function in this case is, from Eq. (4b),

$$\Gamma_{12}(\omega) = \frac{1}{1-\cos\beta}\int_0^\beta J_0(\bar{\omega}\sin\theta)\sin\theta d\theta, \tag{27a}$$

where, on account of the symmetry of the noise field, the contribution from the lower lobe is the same as from the upper, allowing the integral to be taken over the interval $0 \le \theta \le \beta$. By making the substitution $y = \cos\theta$, Eq. (27a) reduces to

$$\Gamma_{12}(\omega) = \frac{1}{1-\cos\beta}\int_{\cos\beta}^1 J_0\left(\bar{\omega}\sqrt{1-y^2}\right)dy, \tag{27b}$$

where, with $\beta < \pi/2$, the integral cannot be expressed explicitly. A numerical integration, however, is easy to compute, leading to the curve shown in Fig. 8(a) for the case $\beta = \pi/4$.

The cross-correlation function may be treated in much the same way as that for the case of the horizontal noise lobe [Eq. (24a)], which leads to the result

$$\psi_{12}(\tau) = \frac{S_0}{2\tau_d(1-\cos\beta)}[u(\tau+\tau_d\sin\beta) - u(\tau-\tau_d\sin\beta)] \times \left\{1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\cos\beta}{\sqrt{1-(\tau/\tau_d)^2}}\right)\right\}. \tag{28}$$

When $\beta = \pi/2$, this expression reduces correctly to the form for isotropic noise in Eq. (17b). In general, the correlation window in Eq. (28) extends over the interval $-\tau_d \sin\beta \le \tau \le \tau_d \sin\beta$, the outer limits of which are dictated by the noise wave fronts with the longest travel time between the two sensors. The expression for the cross-correlation function in Eq. (28) is plotted in Fig. 8(b) for the case $\beta = \pi/4$.

By straightforward differentiation, the derivative of the cross-correlation function with respect to delay time is found to be

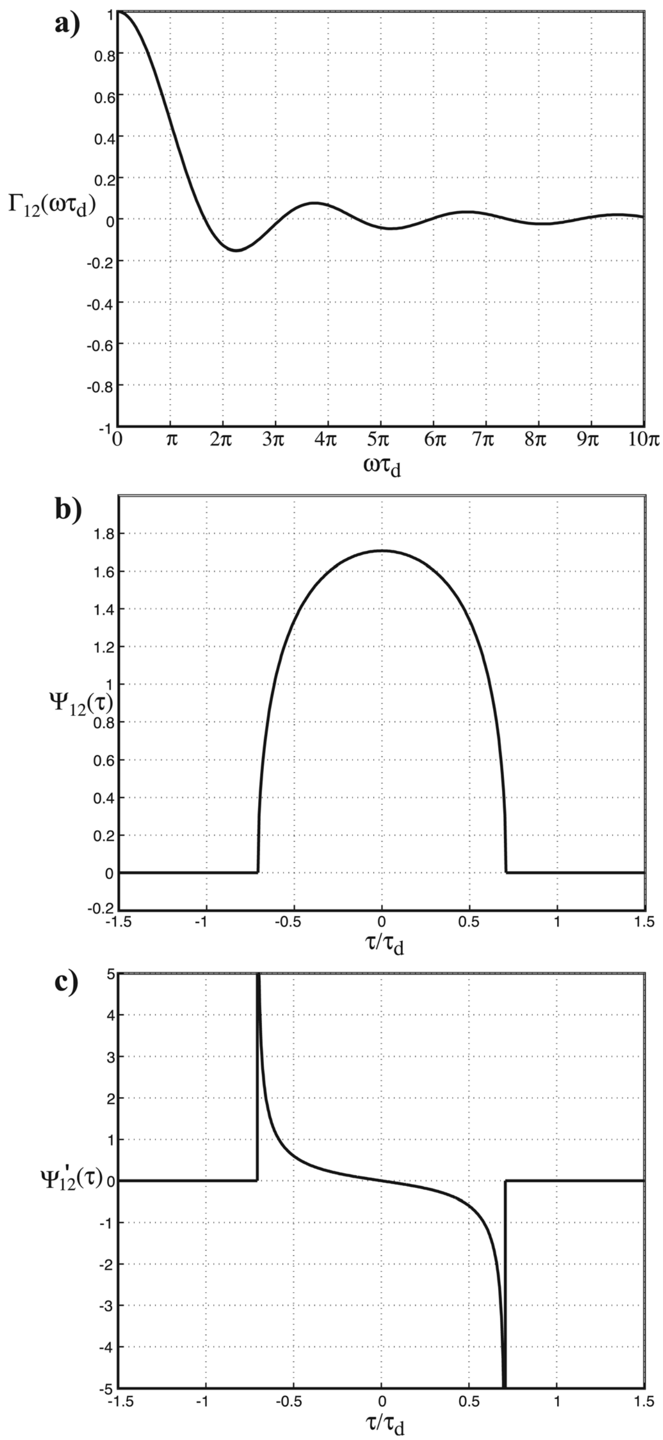

FIG. 8. Horizontal noise notch, $\beta = \pi/4$ and horizontally aligned sensors: (a) coherence function evaluated from Eq. (27b); (b) normalized cross-correlation function from Eq. (28); (c) derivative of the normalized cross-correlation function from Eq. (29b).

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d(1-\cos\beta)}\left[[\delta(\tau+\tau_d\sin\beta) - \delta(\tau-\tau_d\sin\beta)] \times \left\{1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\cos\beta}{\sqrt{1-(\tau/\tau_d)^2}}\right)\right\} - [u(\tau+\tau_d\sin\beta) - u(\tau-\tau_d\sin\beta)]\frac{2}{\pi\tau_d}\cos\beta \times \frac{(\tau/\tau_d)}{\{1-(\tau/\tau_d)^2\}\sqrt{\sin^2\beta - (\tau/\tau_d)^2}}\right]. \quad (29a)$$

For $\beta < \pi/2$, the term involving the delta functions is identically zero and Eq. (29a) reduces to

$$\psi'_{12}(\tau) = -\frac{S_0}{2\tau_d(1-\cos\beta)}[u(\tau+\tau_d\sin\beta) - u(\tau-\tau_d\sin\beta)]\frac{2}{\pi}\cos\beta \times \frac{(\tau/\tau_d^2)}{[1-(\tau/\tau_d)^2]\sqrt{\sin^2\beta-(\tau/\tau_d)^2}}, \quad \text{for } \beta < \pi/2. \quad (29b)$$

When $\beta = \pi/2$, representing isotropic noise, the term in parenthesis multiplying the delta functions is unity and the second term on the right of Eq. (29a) is zero, leading to the result

$$\psi'_{12}(\tau) = \frac{S_0}{2\tau_d}[\delta(\tau+\tau_d) - \delta(\tau-\tau_d)] \quad \text{for } \beta = \pi/2, \quad (29c)$$

which is clearly correct.

Evidently, in the presence of a noise notch of finite angular width, however narrow, the derivative of the cross-correlation function, as given by Eq. (29b), exhibits no delta functions at $|\tau| = \tau_d$, consistent with the absence of endfire components in the noise field. Nor are there delta functions at $|\tau| = \tau_d \sin\beta$, the time delays corresponding to the step-function discontinuities in the noise directional density function. The functional form of Eq. (29b) is shown in Fig. 8(c) for the case $\beta = \pi/4$, where it can be seen that as $|\tau|$ approaches $\tau_d \sin\beta$ from below, the derivative of the cross-correlation function asymptotes to plus or minus infinity. With $\beta < \pi/2$, the divergence of $\psi'_{12}(\tau)$ illustrated in Fig. 8(c) does not occur at the retarded time and anyway is distinct from the behavior of a delta function. The inevitable conclusion is that neither the cross-correlation function nor its derivative can be identified with the free-space Green's function.

VI. DEEP-WATER AMBIENT NOISE

Although it is instructive to examine the discontinuous anisotropic noise fields considered above, they are at best only crudely representative of ambient noise fields in the ocean. In an effort to develop a more realistic representation, Cron and Sherman[45,46] proposed a model of deep-water ambient noise in which independent point sources are distributed uniformly (statistically) in a horizontal plane immediately beneath the sea surface, and the ocean itself is taken to be a semi-infinite, homogeneous half-space. Thus, in Cron and Sherman's representation,[45,46] the noise is azimuthally uniform and travels downwards, with no upward-traveling components. This lack of symmetry about the horizontal gives rise to a coherence function between the fluctuations at vertically aligned sensors that is complex. Of course, the coherence function between the fluctuations at horizontally aligned sensors remains real.

Cron and Sherman's noise field[45,46] consists of two components, one spatially homogeneous and the other evanescent.[47–49] The latter is significant only within a few wavelengths of the surface sources and was, in fact, omitted by Cron and Sherman[45,46] from their analysis. It is also neglected here on the assumption that the sensor station is many wavelengths below the sea-surface sources.

Each noise source and its negative image in the pressure-release sea surface acts as a dipole with power directivity function $g^2(\varphi)$, where φ is the angle measured from the downward vertical. At great depth, such a source distribution gives rise to a plane-wave noise field with a directional density function

$$F(\theta) = \left[u(\theta) - u\left(\theta - \frac{\pi}{2}\right)\right]\frac{g^2(\theta)}{\cos\theta}F_2, \quad (30a)$$

where F_2 is a constant. Since the power directivity of a dipole is given by $g^2(\varphi) = \cos^2\varphi$, the directional density function becomes

$$F(\theta) = \left[u(\theta) - u\left(\theta - \frac{\pi}{2}\right)\right]4\cos\theta, \quad (30b)$$

where F_2 has been evaluated from the normalization condition in Eq. (1). In fact, Eq. (30b) is a special case of the more general situation considered by Cron and Sherman[45,46] in which $g(\varphi) = \cos^m\varphi$, where m is a positive integer. Cron and Sherman's[45,46] angular distribution for dipole sources in Eq. (30b) has been used by Harrison and Siderius[50] in connection with beam-to-beam cross-correlation of ambient noise and by Siderius *et al.*[51] in an analysis of the passive fathometer, a shallow-water technique whereby sub-bottom properties are recovered from ambient noise recorded on a vertical hydrophone array.

A. Vertical sensors

The coherence function of the noise at two vertically aligned sensors is, from Eq. (4a),

$$\Gamma_{12}(\omega) = 2\int_0^{\pi/2} e^{-i\omega\tau_d\cos\theta}\cos\theta\sin\theta d\theta$$
$$= 2\left[\frac{ie^{-i\omega\tau_d}}{\omega\tau_d} + \frac{e^{-i\omega\tau_d}-1}{\omega^2\tau_d^2}\right]$$
$$= 2\left[\frac{\sin\omega\tau_d}{\omega\tau_d} + \frac{\cos\omega\tau_d - 1}{\omega^2\tau_d^2}\right]$$
$$+ 2i\left[\frac{\cos\omega\tau_d}{\omega\tau_d} - \frac{\sin\omega\tau_d}{\omega^2\tau_d^2}\right], \quad (31)$$

which, in effect, is Cron and Sherman's result,[46] although they refer to it as a "spatial correlation function." The coherence function in Eq. (31) is shown in Fig. 9(a).

Unlike the cases considered previously, the coherence function in Eq. (31) is complex, reflecting the fact that the noise field is not symmetrical about the horizontal. As always with vertically aligned sensors in an azimuthally uniform noise field, the real and imaginary parts, respectively, of Eq. (31) are associated with the symmetric and anti-symmetric components of the noise field about the horizontal. To identify these components, Eq. (31) is written in the form

$$F(\theta) = 2|\cos\theta| + 2|\cos\theta|\mathrm{sgn}(\cos\theta), \quad (32)$$

where $0 \le \theta \le \pi$, sgn($\cdots$) is the signum function, and the symmetric and anti-symmetric components, respectively, are the first and second terms on the right.

The cross-correlation function of the noise field is

$$\psi_{12}(\tau) = \frac{S_0}{\pi}\int_{-\infty}^{\infty}\left[\frac{ie^{-i\omega\tau_d}}{\omega\tau_d} + \frac{e^{-i\omega\tau_d}-1}{\omega^2\tau_d^2}\right]e^{i\omega\tau}d\omega, \quad (33a)$$

which evaluates to

$$\psi_{12}(\tau) = [u(\tau) - u(\tau-\tau_d)]\frac{2S_0\tau}{\tau_d^2}; \quad (33b)$$

and the derivative of this function with respect to the delay time, τ, is

$$\psi'_{12}(\tau) = -\frac{2S_0}{\tau_d}\left\{\delta(\tau-\tau_d) - \frac{1}{\tau_d}[u(\tau)-u(\tau-\tau_d)]\right\}. \quad (34)$$

All the integrals needed to reach the result in Eq. (33b) are standard forms that may be interpreted in terms of the familiar properties of generalized functions. The cross-correlation function and its derivative, Eqs. (33b) and (34), are shown in Figs. 9(b) and 9(c), respectively. The absence of correlation for all negative time delays is consistent with the fact that there are no upward-traveling components of the noise field.

In a semi-infinite, isotropic ocean with a pressure-release surface, the Green's function for downward propagation between two vertically aligned source-receiver positions is

$$g(t) = \frac{1}{d}\delta(t-\tau_d) - \frac{1}{(d+2z_1)}\delta\left(t-\tau_d-\frac{2z_1}{c}\right), \quad (35)$$

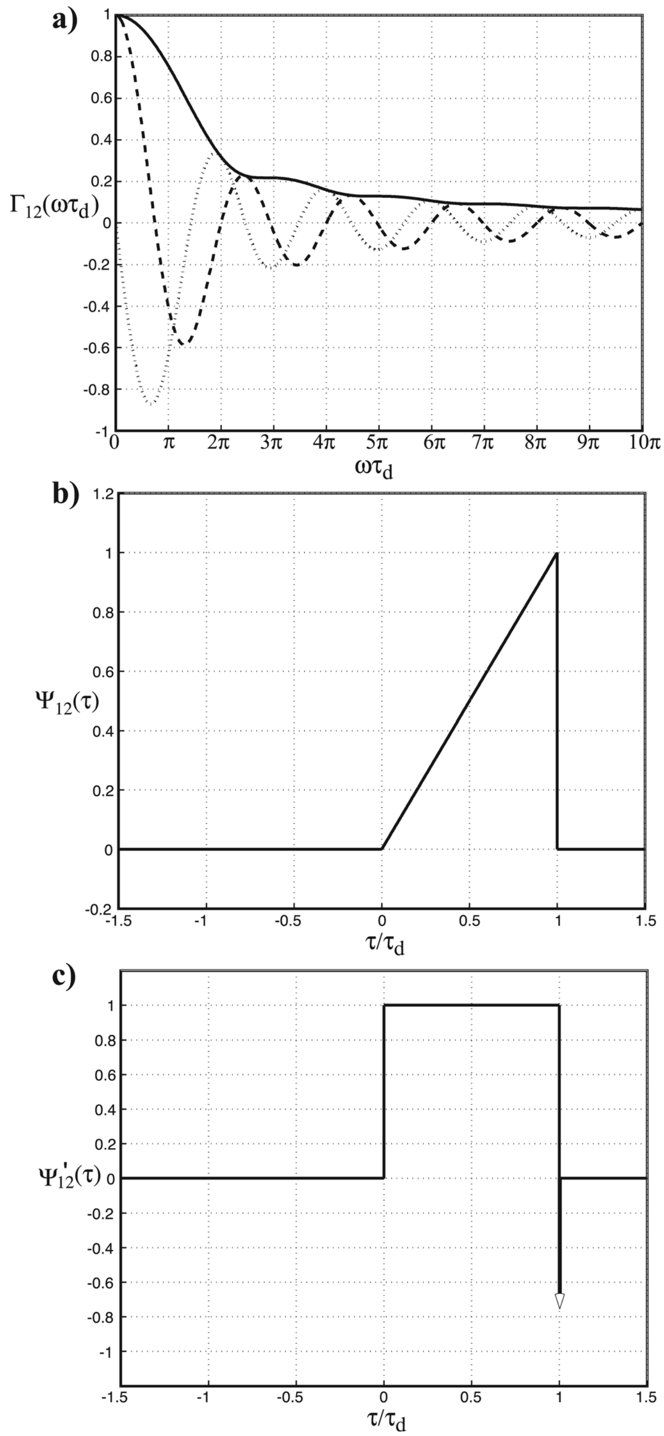

FIG. 9. Deep-water noise and vertically aligned sensors: (a) real (dashed line), imaginary (dotted line), and absolute value (solid line) of the coherence function from Eq. (31); (b) normalized cross-correlation function from Eq. (33b); (c) its derivative with respect to time delay from Eq. (34).

where z_1 is the depth of the impulsive source (taken to be at the top receiver position). The first term on the right of Eq. (35) represents direct-path propagation from the source to the lower receiver, while the second term is the surface-reflected arrival. On comparing Eqs. (34) and (35), it is evident that the derivative of the cross-correlation function of the noise contains a term [the delta function represented by the open arrow in Fig. 9(c)] corresponding to the direct arrival in the Green's function but there is no term that is representative of the surface-reflection. Moreover, the second term in parenthesis in Eq. (34) has no analogue in the Green's function. The absence of a surface-reflected term in Eq. (34) is hardly surprising, since such a term would necessarily involve the depth of the sensors (i.e., z_1 in the Green's function), but that is not possible in a spatially homogeneous noise field, the statistical properties of which are independent of absolute position in the field.

B. Horizontal sensors

From Eq. (4b), the coherence function of the noise at two horizontally aligned sensors is

$$\Gamma_{12}(\omega) = 2\int_0^{\pi/2} J_0(\omega\tau_d \sin\theta)\cos\theta\sin\theta d\theta = \frac{2}{\omega\tau_d}J_1(\omega\tau_d), \tag{36}$$

where $J_1(\cdots)$ is the Bessel function of the first kind of order unity and the integral is a standard form that is available from tables of integrals.[52] As with Eq. (31), the result in Eq. (36), plotted in Fig. 10(a), is familiar from the work of Cron an Sherman.[45]

The cross-correlation function follows from Eq. (7), which yields

$$\psi_{12}(\tau) = \frac{S_0}{\pi\tau_d}\int_{-\infty}^{\infty} J_1(\omega\tau_d)e^{i\omega\tau}\frac{d\omega}{\omega} = \frac{2S_0}{\pi\tau_d}\int_0^{\infty} J_1(\omega\tau_d)\cos\omega\tau\frac{d\omega}{\omega}. \tag{37}$$

The second integral here is a known form,[53] which leads to the result

$$\psi_{12}(\tau) = \frac{2S_0}{\pi\tau_d}[u(\tau+\tau_d) - u(\tau-\tau_d)]\cos[\sin^{-1}(\tau/\tau_d)], \tag{38a}$$

and on differentiating this expression with respect to the delay time, τ, the derivative of the cross-correlation function is found to be

$$\psi'_{12}(\tau) = -\frac{2S_0}{\pi\tau_d^2}[u(\tau+\tau_d) - u(\tau-\tau_d)]\frac{\tau/\tau_d}{\sqrt{1-(\tau/\tau_d)^2}}. \tag{38b}$$

Equations (38a) and (38b) are plotted in Figs. 10(b) and 10(c), respectively.

Although $\psi'_{12}(\tau)$ does diverge to infinity at either end of the correlation window, at $\tau = \pm\tau_d$, these two infinities are not delta functions. They are anti-symmetric, arising from the radical in the denominator of Eq. (38b). Clearly, the infinities in Fig. 10(c) bear a resemblance to the delta functions associated with isotropic noise in Fig. 2(c) but strictly are distinct from the free-space Green's function.

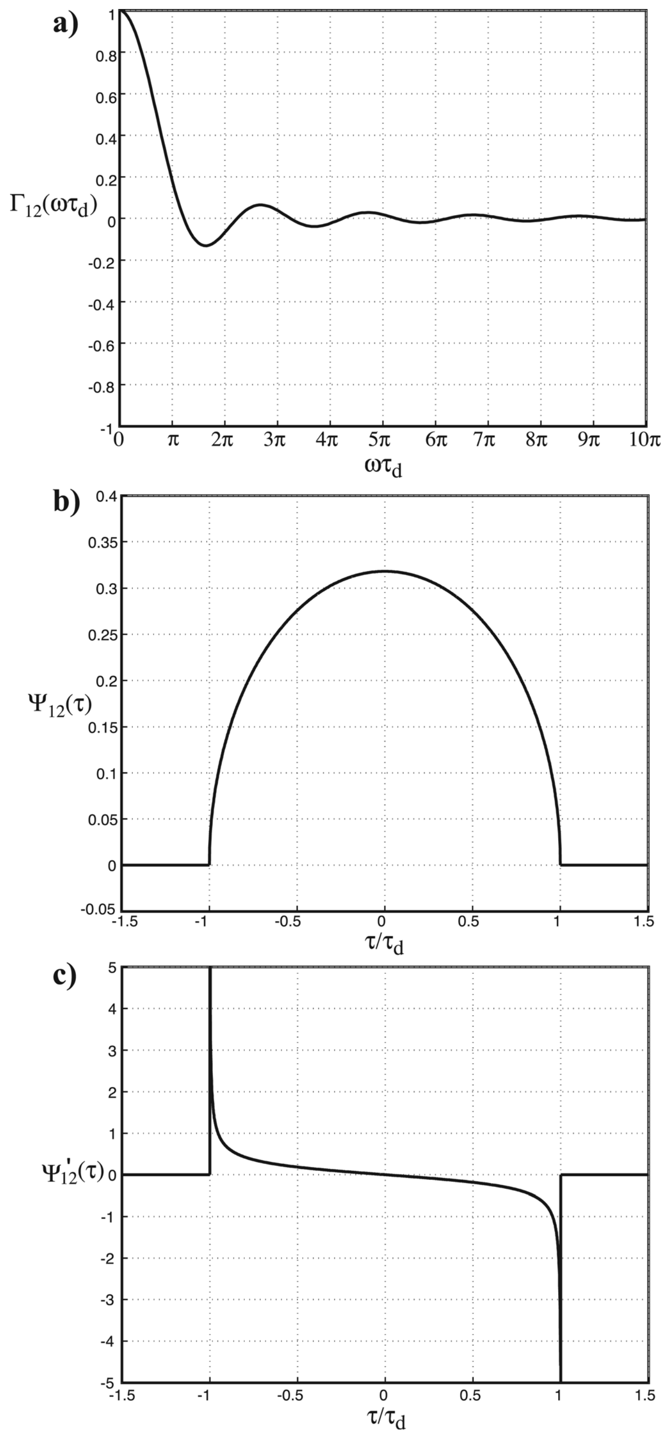

FIG. 10. Deep-water noise and horizontally aligned sensors: (a) coherence function from Eq. (36); (b) normalized cross-correlation function from Eq. (38a); (c) its derivative with respect to delay time from Eq. (38b).

Of course, for a semi-infinite ocean the Green's function is not the free-space version in Eq. (18a), but rather it consists of two delta functions, taking the form

$$g(t) = -\frac{1}{d}\delta(t-\tau_d) + \frac{1}{d\sqrt{1+(2z_1/d)^2}} \times \delta\left(t - \tau_d\sqrt{1+(2z_1/d)^2}\right), \tag{39}$$

where z_1 is the depth of the source/receiver pair. The first term on the right represents the direct-path arrival and the second term the surface-reflected arrival from the negative image of the source in the pressure-release surface. No negative-image component is present in the cross-correlation function of the noise [Eq. (38a)] or in its derivative with respect to delay time [Eq. (38b)]. In fact, the appearance of a negative-image component in the cross-correlation function or its derivative is impossible, since it would necessarily depend on the depth of the sensors in the water column. But Cron and Sherman's noise field[45,46] is spatially homogeneous, and hence its statistical properties are independent of absolute position in the field. A term representative of a surface-reflection cannot, therefore, appear in any of the statistical measures of the noise.

VII. GREEN'S FUNCTION FROM NOISE

Although the free-space Green's function may be recovered from the derivative (with respect to time delay) of the cross-correlation function of isotropic noise, this is not generally the case for anisotropic noise fields. Isolated delta functions in $\psi'_{12}(\tau)$ that may be identified with the free-space Green's function appear only when the directional density function satisfies certain conditions. Since these conditions depend on the orientation of the sensor pair, it is convenient to discuss vertically and horizontally aligned receivers separately.

A. Vertical sensors

The general relationship between the cross-correlation function and the directional density function is given by Eq. (12a). As the directional density function exists only over the angular interval from 0 to π, it may be expressed in the form

$$F(\theta) = [u(\theta) - u(\theta - \pi)]f(\theta), \tag{40}$$

where $f(\theta)$ is a regular function, although it may contain step discontinuities. Bearing in mind that $\cos^{-1}(\tau/\tau_d)$ is monotonic decreasing in the interval $-\tau_d \leq \tau \leq \tau_d$, it follows that the cross-correlation function in Eq. (12a) may be written as

$$\begin{aligned}\psi_{12}(\tau) &= \frac{S_0}{2\tau_d}[u(\cos^{-1}(\tau/\tau_d)) - u(\cos^{-1}(\tau/\tau_d) - \pi)] \\ &\quad \times f(\cos^{-1}(\tau/\tau_d)) \\ &= \frac{S_0}{2\tau_d}[u(\tau + \tau_d) - u(\tau - \tau_d)]f(\cos^{-1}(\tau/\tau_d)).\end{aligned} \tag{41}$$

Taking the derivative of this expression with respect to time delay yields

$$\begin{aligned}\psi'_{12}(\tau) = \frac{S_0}{2\tau_d}\Bigg\{&[\delta(\tau+\tau_d) - \delta(\tau-\tau_d)]f\left(\cos^{-1}\left\{\frac{\tau}{\tau_d}\right\}\right) \\ &+ [u(\tau+\tau_d) - u(\tau-\tau_d)]\frac{df(\cos^{-1}\{\tau/\tau_d\})}{d\tau}\Bigg\},\end{aligned} \tag{42}$$

which, with

$$\alpha = \cos^{-1}(\tau/\tau_d), \tag{43}$$

reduces to

$$\begin{aligned}\psi'_{12}(\tau) = \frac{S_0}{2\tau_d}\Bigg\{&[\delta(\tau + \tau_d)f(\pi) - \delta(\tau - \tau_d)f(0)] \\ &- [u(\tau + \tau_d) - u(\tau - \tau_d)] \\ &\times \left(\tau_d\sqrt{1 - (\tau/\tau_d)^2}\right)^{-1}\frac{df(\alpha)}{d\alpha}\Bigg\}.\end{aligned} \tag{44}$$

It is evident from Eq. (44) that anti-symmetric delta functions will appear at $\tau = \pm\tau_d$ only if $f(0)$ and $f(\pi)$ are non-zero, or equivalently, provided that noise rays propagate in both directions along the line connecting the sensors, a condition that has been identified by several previous authors.[7,16,33] If these delta functions are to be the only contributors to $\psi'_{12}(\tau)$, the second term on the right of Eq. (44) must be zero, that is to say, the gradient of the directional density function must be zero everywhere within the angular interval $[0,\pi]$. Only isotropic noise satisfies this zero-gradient condition. For this case, as illustrated in Fig. 2(c), $\psi'_{12}(\tau)$ exhibits two anti-symmetric delta functions at $\tau = \pm\tau_d$ and elsewhere is identically zero. These delta functions can be identified with the free-space Green's function for acoustic propagation from one receiver position to the other.

In principle, it should be possible to recover the free-space Green's function not only from isotropic noise but also from certain anisotropic noise fields. Take, for instance, the class of noise fields characterized by delta functions in $\psi'_{12}(\tau)$ at $\tau = \pm\tau_d$ but with some non-zero structure between them. Suppose that for the purpose of recovering the Green's function, the delta functions are required to be isolated from the intervening structure, unlike the situation in Fig. 9(c), where a negative delta function hangs under a pedestal. If isolated delta functions are to be present in $\psi'_{12}(\tau)$ at $\tau = \pm\tau_d$, it follows from Eq. (44) that $f(0)$ and $f(\pi)$ must be positive, just as in the case of isotropic noise, and in addition the gradients of the directional density function at either end of the angular range must be zero, that is, $f'(0+) = f'(\pi-) = 0$, where the prime denotes differentiation with respect to the argument.

A simple example of such behavior is provided by the noise notch, as shown in Fig. 7(c), where the interior structure of $\psi'_{12}(\tau)$ takes the form of two anti-symmetric delta functions located at $\tau = \pm\tau_d \cos\beta$. These delta functions arise from the step-function discontinuities in the directional density function at $\theta = \beta$ and $\theta = \pi - \beta$. In general, if a step-function discontinuity appears in the directional density function at say, $\theta = \theta_0$, then an associated delta function will be present in $\psi'_{12}(\tau)$ at $\tau = \tau_d \cos\theta_0$. This delta function will be isolated if $f'(\theta_0-) = f'(\theta_0+) = 0$ but otherwise will sit on a pedestal whose shape is a mapping of the directional density function itself.

B. Horizontal sensors

The situation is somewhat different with horizontal sensors and a little more complicated because of the integral formulation of the cross-correlation function in Eq. (12b). Taking the second of the integrals in Eq. (12b) and differentiating with respect to the time delay, it can be shown, after some algebraic manipulation, that

$$\psi'_{12}(\tau) = \frac{S_0}{2\pi\tau_d}\left\{[\delta(\tau+\tau_d)-\delta(\tau-\tau_d)]\pi F\left(\frac{\pi}{2}\right) + [u(\tau+\tau_d)-u(\tau-\tau_d)]\frac{(\tau/\tau_d)}{\tau_d[1-(\tau/\tau_d)^2]} \times \int_{\theta_1}^{\pi-\theta_1}\frac{\cos\alpha F'(\alpha)}{\sqrt{\sin^2\alpha-\sin^2\theta_1}}d\alpha\right\}, \quad (45)$$

where the prime denotes a derivative with respect to the argument and θ_1 is the function of the correlation delay time defined in Eq. (13). It follows from the term containing the anti-symmetric delta functions in Eq. (45) that the free-space Green's function can be recovered from $\psi'_{12}(\tau)$ provided that $F(\pi/2)$ is non-zero. This is the familiar condition[7,16,33] that, if delta functions are to appear in $\psi'_{12}(\tau)$ at $\tau = \pm\tau_d$, horizontal noise components must be present in the field, since these are the components that propagate along the line connecting the two sensors.

If there are no horizontal noise components in the field, that is, $F(\pi/2) = 0$, the delta functions at $\tau = \pm\tau_d$ vanish, as exemplified by Cron and Sherman's deep-water noise field [Fig. 10(c)].[45,46] Moreover, in the absence of horizontal noise, no delta functions will appear anywhere in $\psi'_{12}(\tau)$, even though the directional density function may exhibit step-function discontinuities [e.g., Fig. 8(c) for the noise notch]. This differs from the case of vertical sensors, where a step-function discontinuity in $F(\theta)$ at $\theta = \chi$ gives rise to a delta function in $\psi'_{12}(\tau)$ at $\tau = \tau_d \cos\chi$ [e.g., Fig. 4(c) for the horizontal noise lobe].

VIII. CONCLUDING REMARKS

In this article, an analysis of the vertical and horizontal two-point spatial statistics of three types of azimuthally uniform, vertically anisotropic, ocean ambient noise field is presented. Two of the noise fields considered are somewhat idealized, one showing a horizontal lobe and the other a horizontal notch, while the third is a more realistic representation of deep-water ambient noise, as created by a random distribution of monopole sources in a horizontal plane immediately beneath the sea surface. All three of these noise fields are spatially homogeneous, consisting of a random superposition of plane-waves propagating in all directions. In each case, the coherence function, the cross-correlation function and the derivative of the latter with respect to the correlation delay time have been derived for two sensor orientations, vertical and horizontal. The main focus of interest is the derivative (with respect to time delay) of the cross-correlation function, since this derivative may or may not contain delta functions that can be identified with the free-space Green's function.

As is well established, anti-symmetric delta functions will appear in the derivative of the cross-correlation function at a correlation delay equal to the retarded time provided significant noise components propagate along the line connecting the two sensors. This is true regardless of the orientation of the sensors in the field. These delta functions have been identified with the free-space Green's function by several previous authors.[7,16,33]

With vertically aligned sensors, if the directional density function of the noise contains a step-function discontinuity at polar angle $\theta = \chi$, a delta function will appear in the derivative of the cross-correlation function at delay time $\tau = \tau_d \cos\chi$, where τ_d is the retarded time. Isotropic noise has step-function discontinuities at $\theta = 0$ and $\theta = \pi$, consistent with the much discussed anti-symmetric delta functions at $\tau = \pm\tau_d$ in the derivative of its cross-correlation function. More generally, any delta function appearing in the derivative of the cross-correlation function will be isolated, that is, not sitting on a pedestal, provided the derivative of the directional density function is zero in the vicinity of the associated step-function discontinuity. Otherwise, a pedestal will be present whose shape is a mapping of the directional density function itself.

In the case of horizontally aligned sensors, the only delta functions that can appear in the derivative of the cross-correlation function are associated with horizontally traveling noise, that is, noise traveling along the line connecting the two receivers. If they exist, such delta functions always appear at the retarded time on either side of the origin. Even if the directional density function contains a step-function discontinuity in a direction other than the horizontal, it does not map into a delta function in the derivative of the cross-correlation function. It is possible for infinities to occur, but since these are distinct from delta functions they cannot be identified with the free-space Green's function.

ACKNOWLEDGMENTS

This research was supported by the Office of Naval Research, Ocean Acoustics Code 321, under Grant Number N00014-10-1-0092, sponsor Dr. Robert Headrick.

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