On the transient solutions of three acoustic wave equations: van Wijngaarden's equation, Stokes' equation and the time-dependent diffusion equation

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Acoustic wave propagation in a dispersive medium may be described by a wave equation containing one or more dissipation terms. Three such equations are examined in this article: van Wijngaarden's equation (VWE) for sound propagating through a bubbly liquid; Stokes' equation for acoustic waves in a viscous fluid; and the time-dependent diffusion equation (TDDE) for waves in the interstitial gas in a porous solid. The impulse-response solution for each of the three equations is developed and all are shown to be strictly causal, with no arrivals prior to the activation of the source. However, the VWE is nonphysical in that it predicts instantaneous arrivals, which are associated with infinitely fast, propagating Fourier components in the Green's function. Stokes' equation and the TDDE are well behaved in that they do not predict instantaneous arrivals. Two of the equations, the VWE and Stokes' equation, satisfy the Kramers-Kronig dispersion relations, while the third, the TDDE, does not satisfy Kramers-Kronig, even though its impulse-response solution is causal and physically realizable. The Kramers-Kronig relations are predicated upon the (mathematical) existence of the complex compressibility, a condition which is not satisfied by the TDDE because the Fourier transform of the complex compressibility is not square-integrable.
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I. INTRODUCTION

Any partial differential equation describing acoustic-wave propagation in a dispersive medium contains at least one term that is representative of dissipation in the material. Three such equations are examined in this article, all of which are based on the assumption that the medium supporting the propagation is a homogeneous, time-invariant continuum: van Wijngaarden's equation[1] (VWE) for acoustic waves in an isothermal, viscous, bubbly liquid; Stokes' equation,[2] which may be thought of as a special case of the VWE, for acoustic waves in a bubble-free, viscous fluid; and the time-dependent diffusion equation[3] (TDDE) for waves in the interstitial gas filling a porous, statistically isotropic, perfectly rigid solid.[4] In all three cases, the solutions of these equations for harmonic waves have long been known; but the transient solutions are less amenable to analysis. As a result, a number of misconecptions have appeared in the literature. For instance, Jordan *et al.*[5] claim that Stokes' equation yields solutions that do not satisfy causality; and Blackstock[6] concludes that Stokes' equation is nonphysical, based on the appearance of an infinitely fast precursor in his solution. In fact, as demonstrated later, the transient solutions of Stokes' equation[7] are perfectly well-behaved: (1) they satisfy causality, with no arrivals present prior to the activation of the source; and (2) they are physically realizable, with no instantaneous arrivals predicted anywhere in the medium.

Much of the of the interest in the transient solutions of wave equations focuses on causality and the behavior of the predicted arrival around $t=0$, the time at which the source is activated. Although extensive discussions of causality may be found in the literature,[8–10] a simple definition holds true: a macroscopic system is said to be causal provided that the output does not precede the input. Similarly, a wave equation may be said to be causal provided that, everywhere in the medium, its solutions are zero prior to the time at which the source is activated.

Although causality requires that the effect must not precede the cause, this condition is not sufficient to ensure that the system, or predicted response in the case of an equation, is physically realizable. Since an infinitely fast propagating wave is impossible, at least in the realm of continuum mechanics, the first nonzero arrival at a receiver that is not coincident with the source must appear at some finite time after the activation of the source. In other words, instantaneous arrivals are nonphysical. Thus, to be physical, the first nonzero response must be delayed relative to the activation time of the source.

Assuming that they are accessible, the transient solutions reveal whether a wave equation is not only causal but also physical. Based on such solutions, it will be shown that the VWE is causal but not physical, while Stokes' equation and the TDDE are both causal and physical. As an alternative to the transient solutions, it is possible to investigate causality, but not physical realizability, through the Kramers-Kronig dispersion relations,[11,12] which were developed originally in connection with optical wave propagation in a

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dielectric material. Dispersion relations, which take the form of an integral-transform pair, express the real part of a complex function, $K(\omega)$, in terms of the Hilbert transform of the imaginary part and *vice versa*. In acoustics, the function $K(\omega)$ represents the complex compressibility, which scales inversely with the square of the complex sound speed; and the real variable ω represents angular frequency. The first investigation of dispersion relations for acoustic waves is attributed to Ginzberg[13] with subsequent contributions from Mangulis,[14] Weaver and Pao,[15] and Lee *et al.*[16]

The Kramers-Kronig dispersion relations hold provided that the physical system under consideration is both linear and causal (certain Fourier-transform existence conditions must also be satisfied). They do not depend on the detailed physical mechanisms underlying the wave propagation and thus they provide less information about the wave field than the transient solutions of the associated wave equation. Notwithstanding, the Kramers-Kronig dispersion relations have traditionally provided a useful test of causality. Thus, a wave equation that possesses causal transient solutions might be expected to satisfy Kramers-Kronig, whereas a wave equation that does not satisfy Kramers-Kronig might be expected to return noncausal transient solutions. It turns out that neither of these expectations is necessarily correct: all three of the wave equations investigated in this article are causal but only two, the VWE and Stokes' equation, satisfy Kramers-Kronig. The TDDE does not satisfy Kramers-Kronig but nevertheless is perfectly well behaved in that its impulse response is not only causal but also physically realizable. The VWE, on the other hand, satisfies Kramers-Kronig and is causal but returns an impulse response that is nonphysical. Stokes' equation also satisfies Kramers-Kronig, is causal and, unlike the VWE, yields an impulse response that is physically realizable.

Before pursuing the transient solutions of the three wave equations, it is instructive to develop the Kramers-Kronig relations for acoustic waves in a dispersive medium. This is achieved by following an argument similar to that of O'Donnell *et al.*,[17] which is predicated upon (1) linear acoustic propagation, implying that superposition holds, and (2) a causal relationship between the pressure and density fluctuations in the medium.

II. ACOUSTIC DISPERSION RELATIONS

A. Derivation of the Kramers-Kronig relations

As a sound wave propagates through a dispersive medium, the pressure fluctuation is accompanied by a change in the density. Provided both are sufficiently small, as is usually the case for acoustic waves, the pressure and density fluctuations may be regarded as the input and output, respectively, of a linear system whose impulse response is the time-domain compressibility $k(t)$. Thus, the equation of state may be expressed in the time domain as a convolution integral having the form

$$r(t) = \int_{-\infty}^{\infty} k(t-t')p(t')dt', \tag{1}$$

where $p(t)$ is the pressure fluctuation about the mean and $r(t)$ is the relative density fluctuation, or condensation,[18]

$$r(t) = \frac{\rho(t)-\rho_0}{\rho_0} \ll 1. \tag{2}$$

In this expression, $\rho(t)$ is the absolute density fluctuation and ρ_0 is the bulk density of the medium. Since $k(t)$ is a measurable quantitiy, it is a real function of time.

Assuming that the Fourier transforms of $r(t)$, $k(t)$, and $p(t)$ exist, that is, the time functions are square-integrable, then the convolution in Eq. (1) may be expressed in the frequency domain as the product

$$R(\omega) = K(\omega)P(\omega), \tag{3}$$

where the uppercase letters denote the Fourier transforms of the corresponding time functions

$$R(\omega) = \int_{-\infty}^{\infty} r(t)e^{-i\omega t}dt, \tag{4}$$

and similarly for $K(\omega)$ and $P(\omega)$. Now, on taking the pressure in Eq. (1) to be an impulse,

$$p(t) = q_0\delta(t), \tag{5}$$

where q_0 is a constant and $\delta(\cdots)$ is the Dirac delta function, and performing the appropriate inverse Fourier transform, the condensation becomes

$$r(t) = q_0 k(t) = \frac{q_0}{2\pi}\int_{-\infty}^{\infty} K(\omega)e^{i\omega t}d\omega. \tag{6}$$

At this point in the argument, the concept of causality is introduced. The condensation must be zero prior to the onset of the pressure at $t=0$ and hence the Fourier integral in Eq. (6) has to be zero for negative times. This condition may be written as

$$\int_{-\infty}^{\infty} K(\omega')e^{-i\omega' t}d\omega' = 0 \quad \text{for } t > 0. \tag{7}$$

By allowing ω' to become complex, and assuming that the frequency-domain compressibility converges according to the condition

$$\lim_{|\omega'|\to\infty} |K(\omega')| = 0, \quad 0 \geqslant \arg \omega' > -\pi, \tag{8}$$

then it should be clear, by constructing a semicircle of infinite radius around the lower half-plane, that Eq. (7) will be satisfied provided that $K(\omega')$ is analytic both in the lower half of the complex ω'-plane and on the real axis. This causality requirement, that there be no singularities in $K(\omega')$ on or below the real axis, is central to the derivation of the dispersion relations.

To continue, the expression in Eq. (7) is now multiplied by $e^{i\omega t}$ and integrated over positive t, to obtain

$$\int_{-\infty}^{\infty} K(\omega')d\omega' \int_{0}^{\infty} e^{i(\omega-\omega')t}dt = 0. \tag{9}$$

By using the identity

$$\int_{0}^{\infty} e^{iat}dt = \pi\delta(a) - \frac{1}{ia}, \tag{10}$$

which holds for a real, Eq. (9) reduces to

$$\pi K_1(\omega) - i\pi K_2(\omega) - i\int_{-\infty}^{\infty} \frac{K_1(\omega') - iK_2(\omega')}{\omega' - \omega}d\omega' = 0, \tag{11}$$

where K_1 and $-K_2$ are the real and imaginary parts of $K(\omega)$:

$$K(\omega) = K_1(\omega) - iK(\omega), \tag{12}$$

which serves to define the sign convention used hereafter.

The Kramers-Kronig dispersion relations emerge from Eq. (11) by equating the real and the imaginary parts of the left side to zero, which yields

$$K_1(\omega) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{K_2(\omega')}{\omega' - \omega}d\omega' \tag{13a}$$

and

$$K_2(\omega) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{K_1(\omega')}{\omega' - \omega}d\omega'. \tag{13b}$$

The integrals in Eqs. (13a) and (13b) are taken along the real axis, which contains a simple pole at $\omega' = \omega$. This is accommodated by interpreting each integral as the Cauchy principal value, as denoted by "P" in Eqs. (13a) and (13b). According to these expressions, the real and imaginary parts of the complex compressibility are not independent: the real part, $K_1(\omega)$, may be determined if the imaginary part, $-K_2(\omega)$, is known at all frequencies, and *vice versa*. Moreover, if the function $k(t)$ is causal, then the dispersion relations in Eqs. (13a) and (13b) hold, and conversely, if Eqs. (13a) and (13b) hold, then $k(t)$ is causal, which constitutes a partial statement of Titchmarsh's theorem.[9,19]

B. Conjugate poles of $K_1(\omega)$ and $K_2(\omega)$

While causality requires $K(\omega')$ to have no poles on or below the real axis, this is not so of its real and imaginary parts. Both $K_1(\omega')$ and $K_2(\omega')$ have poles in the lower half-plane, which are conjugates of the poles of $K(\omega')$ above the real axis, and these conjugate poles will contribute to the integrals in Eqs. (13a) and (13b). Their effect may be seen by constructing a closed contour, formed by the real axis and a semicircle of infinite radius in the lower half-plane. Since, from Eq. (8), the integral around the semicircle is zero, it follows from Cauchy's theorem that the integrals along the real axis in Eqs. (13a) and (13b) may be written as

$$K_1(\omega) = -i\left\{K_2(\omega) + 2\sum \text{ residues of } \frac{K_2(\omega')}{\omega' - \omega} \text{ from conjugate poles of } K_2(\omega)\right\}, \tag{14a}$$

$$K_2(\omega) = -i\left\{K_1(\omega) + 2\sum \text{ residues of } \frac{K_1(\omega')}{\omega' - \omega} \text{ from conjugate poles of } K_1(\omega)\right\}, \tag{14b}$$

The first term in each of the parentheses in Eqs. (14) is the residue of the pole on the real axis at $\omega' = \omega$, obtained in the usual way by indenting a small semicircle either above or below the pole. From Eqs. (14), it is readily shown that the following equality holds:

$$\sum \text{ residues of } \frac{K_2(\omega')}{\omega' - \omega} \text{ from conjugate poles of } K_2(\omega) = i\sum \text{ residues of } \frac{K_1(\omega')}{\omega' - \omega} \text{ from conjugate poles of } K_1(\omega). \tag{15}$$

Note that the pole on the real axis at $\omega' = \omega$ does not contribute to this identity.

In effect, Eqs. (14a) and (14b) may be regarded as an alternative formulation of the Kramers-Kronig dispersion relations. They state that $K_1(\omega)$ can be computed from $K_2(\omega)$, and *vice versa*, provided only that the poles of $K(\omega)$ are known. In this case, unlike the situation with the Kramers-Kronig integrals themselves, it would not be necessary to know the real (imaginary) part of $K(\omega)$ at all frequencies in order to compute the imaginary (real) part of $K(\omega)$. However, such an application of Eqs. (14a) and (14b) does present certain difficulties, at least in the context of an attempt to determine, say, $K_1(\omega)$ from experimental measurements of $K_2(\omega)$, since such measurements do not return the required poles. On the other hand, Eqs. (14a) and (14b) provide a useful tool for investigating the causal properties of partial differential wave equations of the type to be discussed later.

C. Wave speed and attenuation

The complex compressibility $K(\omega)$ is related to the complex sound speed $c(\omega)$ as follows:

$$\frac{1}{c^2(\omega)} = \rho_0 K(\omega), \tag{16}$$

which, in terms of the phase speed $c_p(\omega)$ and attenuation $\alpha_p(\omega)$ becomes

$$\left[\frac{1}{c_p(\omega)} - \frac{i\alpha_p(\omega)}{\omega}\right]^2 = \rho_0[K_1(\omega) - iK_2(\omega)]. \tag{17}$$

By equating real part to real part and imaginary part to imaginary part, this may be decomposed into

$$\frac{1}{c_p^2(\omega)} - \frac{\alpha_p^2}{\omega^2} = \rho_0 K_1(\omega) \tag{18a}$$

and

$$\frac{2\alpha_p(\omega)}{\omega c_p(\omega)} = \rho_0 K_2(\omega). \tag{18b}$$

In these expressions, the phase speed and attenuation are coupled, which raises difficulties in using the Kramers-Kronig dispersion relations to compute one from knowledge of the other, even though that knowledge may be exact and for all frequencies. The situation is mitigated if, as is often the case, the attenuation is small such that

$$\frac{\alpha_p(\omega)c_p(\omega)}{\omega^2} \ll 1, \tag{19}$$

for then the phase speed may be written explicitly in terms of $K_1(\omega)$:

$$c_p(\omega) \approx \frac{1}{\sqrt{\rho_0 K_1(\omega)}}. \tag{20a}$$

But $K_2(\omega)$ is still mixed in $c_p(\omega)$ and $\alpha_p(\omega)$:

$$\alpha_p(\omega) = \frac{\rho_0 c_p(\omega)}{2}\omega K_2(\omega). \tag{20b}$$

By substituting these expressions for $K_1(\omega)$ and $K_2(\omega)$ into the Kramers-Kronig dispersion relation in Eq. (13b), it is evident that the attenuation can be computed if the phase speed is known at all frequencies. The converse, however, is not true; the phase speed cannot be derived from knowledge of the attenuation at all frequencies, although O'Donnel *et al.*[17] suggest otherwise. The difficulty is apparent from Eq. (13a), which, with Eqs. (20a) and (20b), becomes

$$\frac{1}{c_p^2(\omega)} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\alpha_p(\omega')}{\omega' c_p(\omega')(\omega' - \omega)} d\omega'. \tag{21}$$

Thus, the unknown function $c_p(\omega)$ occurs both outside and inside the integral, creating an intractable integral equation. Certain integral equations can be solved, but not so in this case, which leads to the conclusion that the wave speed cannot be determined from full knowledge of the attenuation.

III. van WIJNGAARDEN'S EQUATION

In 1972, van Wijngaarden[1] developed a wave equation describing the propagation of acoustic waves in an isothermal, viscous, bubbly liquid. For one-dimensional propagation, the inhomogeneous form of the VWE for the velocity potential, $g = g(x,t)$, is

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 g}{\partial t^2} + \gamma\frac{\partial^3 g}{\partial t \partial x^2} + \beta^2 \frac{\partial^4 g}{\partial t^2 \partial x^2} = -Q\delta(x)\delta(t), \tag{22}$$

where the delta functions on the right represent a planar, impulsive source and Q is the source strength with dimensions of (volume per unit area) = length. On the left, c_0 is the sound speed in the limit of low frequency or, equivalently, in the absence of the dissipation terms; the third term represents viscous effects, with the coefficient γ proportional to the dynamic viscosity of the bubbly mixture; and bubble-size effects are represented by the fourth term with coefficient β^2, which scales as the square of the equilibrium bubble radius. The homogeneous form of Eq. (22) has been examined by Jordan and Feuillade[20] in connection with a boundary value problem.

A comment on semantics is perhaps in order. In the following discussions, when referring to "van Wijngaarden's equation" or the "VWE," it is implied that the bubble-radius coefficient β^2 is greater than zero. When β^2 is equal to zero, representing an absence of bubbles, the fourth term on the left vanishes and Eq. (22) reduces to Stokes' equation. Thus, the term "Stokes equation" hereinafter signifies that β^2 is identically zero. This distinction is important because it underlies the fundamentally different behavior exhibited by the VWE and Stokes' equation.

A. Impulse response

To investigate the properties of its transient solutions, the VWE in Eq. (22) is to be solved using the same technique as that developed by Buckingham[7] in connection with Stokes' equation. Two bilateral Fourier transforms are applied to Eq. (22), the first with respect to time t, and the second with respect to distance x. The respective transform variables, angular frequency ω and wavenumber s, are used as subscripts to denote the transformed field, which is a convenient notation when multiple transforms are employed. Bearing in mind that the transforms of the second derivatives take their usual forms, with radiation conditions ensuring no integrated contributions, the result of the double Fourier transformation is the algebraic equation

$$\begin{aligned} g_{\omega s} &= -\frac{Qc_0^2}{[\omega^2(1+\beta^2 c_0^2 s^2) - i\omega s^2 c_0^2 \gamma - c_0^2 s^2]} \\ &= -\frac{Qc_0^2}{(1+\beta^2 c_0^2 s^2)(\omega - \omega_+)(\omega - \omega_-)}, \end{aligned} \tag{23}$$

where the roots of the quadratic in ω are

$$\omega_\pm = i\frac{s^2\gamma c_0^2}{2(1+\beta^2 c_0^2 s^2)} \pm \frac{sc_0\sqrt{1+(\beta^2 - (\gamma c_0/2)^2)s^2}}{(1+\beta^2 c_0^2 s^2)}. \tag{24}$$

Both of these roots lie in the top half of the complex ω-plane and thus $g_{\omega s}$ is analytic in ω on and below the real ω axis. Taking the inverse Fourier transform of Eq. (23) with respect to ω, yields the wavenumber spectrum

$$g_s(t) = -\frac{Qc_0^2}{2\pi(1+\beta^2 c_0^2 s^2)}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-\omega_+)(\omega-\omega_-)}d\omega. \quad (25)$$

For $t<0$, this integral may be evaluated by forming a closed contour bounded by the real axis and a semicircle of infinite radius around the lower half-plane. As this contour contains no poles, the result is zero, indicating that the field is causal since there are no arrivals before $t=0$. For $t>0$, the appropriate contour is the real axis and a semicircle of infinite radius around the upper half-plane, which contains the poles at $\omega=\omega_\pm$. With the aid of Jordan's lemma and Cauchy's theorem,[3] the complete expression for the wavenumber spectrum of the velocity potential may therefore be written as

$$g_s(t) = u(t)\frac{Qc_0}{sR}\exp\left(-\frac{sc_0\gamma q}{2}t\right)\sin(qRt), \quad (26)$$

where

$$R = \sqrt{1+\left(\beta^2-\frac{\gamma^2}{4}\right)s^2 c_0^2}, \quad (27)$$

$$q = \frac{sc_0}{(1+\beta^2 c_0^2 s^2)}, \quad (28)$$

and $u(t)$ is the Heaviside unit step function. When $\beta=0$, Eq. (26) reduces identically to the corresponding solution of Stokes' equation derived by Buckingham.[7]

As the velocity potential is not a physically measurable quantity, it is convenient at this point to convert to the pressure $p(x,t)$. The wavenumber spectrum of the pressure is the time derivative of Eq. (26):

$$p_s(t) = \rho_0\frac{\partial g_s}{\partial t} = u(t)\frac{\rho_0 Qc_0}{sR}q\exp\left(-\frac{sc_0\gamma qt}{2}\right)\times\left\{R\cos(qRt)-\frac{sc_0\gamma}{2}\sin(qRt)\right\}, \quad (29)$$

where ρ_0 is the bulk density of the medium. It is interesting to note that, for $t=0+$, the wavenumber spectrum of the pressure is nonzero, and this is true for both the VWE ($\beta>0$) and Stokes' equation ($\beta=0$). As will be shown shortly, this step discontinuity at the origin of time in the wavenumber spectrum leads to a nonphysical Green's function for the VWE; but for Stokes' equation, the Green's function is perfectly well behaved.

B. The pressure Green's function

To obtain the pressure Green's function for the VWE, an inverse Fourier transform with respect to s is applied to Eq. (29), which yields

$$p(x,t) = u(t)\frac{\rho_0 Qc_0}{2\pi}\int_{-\infty}^{\infty}\frac{q}{sR}\exp\left(-\frac{sc_0\gamma q}{2}t\right)\left\{R\cos(qRt) - \frac{sc_0\gamma}{2}\sin(qRt)\right\}e^{isx}ds. \quad (30)$$

Obviously, this expression satisfies causality, since the unit step function $u(t)$ ensures that the pressure is zero for $t<0$. It is worth noting that, if the pressure field had been derived by reversing the order of the inverse Fourier transforms applied to Eq. (23), a different formulation from Eq. (30), with no explicit step function at the origin of time, would have been obtained. To be explicit, by inverse transforming Eq. (23) first with respect to wavenumber s, followed by the inverse transform with respect to frequency ω, the pressure Green's function is found to be

$$p(x,t) = \frac{\rho_0 Qc_0}{4\pi}\int_{-\infty}^{\infty}(1+i\omega\gamma-\beta^2\omega^2)^{-1/2}\times\exp\left\{-\frac{i\omega|x|}{c_0\sqrt{1+i\omega\gamma-\beta^2\omega^2}}\right\}e^{i\omega t}d\omega. \quad (31)$$

Of course, Eqs. (30) and (31) are equivalent, since they represent the same impulse response function. In the case of Eq. (31), although there is no step function at the origin, it still returns zero for negative times. This is confirmed by the fact that the integrand has no singularities (poles or branch points) on or below the real axis of the complex ω-plane and hence from Jordan's lemma and Cauchy's theorem, the field from Eq. (31) is identically zero for $t<0$, consistent with Eq. (30).

Both solutions exhibit a step discontinuity everywhere in the medium at $t=0$. The size of the discontinuity can be expressed explicitly from the integral in Eq. (30), which, for the special case $t=0+$, reduces to

$$p(x,0+) = \frac{\rho_0 Qc_0^2}{2\pi}\int_{-\infty}^{\infty}\frac{e^{isx}}{(1+\beta^2 c_0^2 s^2)}ds = \frac{\rho_0 Qc_0}{2\beta}e^{-|x|/\beta c_0} \quad \text{for } \beta>0, \quad (32)$$

a result that may be obtained by elementary contour integration or from tables of integrals.[21] Either way, it is evident that the size of the step depends on the bubble-size coefficient β but not the viscous coefficient γ. The presence of such a step in the pressure at $t=0$ can be attributed only to instantaneous arrivals at the receiver, thus forcing the conclusion that the VWE is nonphysical: such arrivals imply infinitely high wave speeds, which are impossible under the constraints of continuum mechanics. Some insight into the origin of these infinitely fast waves will be gained later from the dispersion curves associated with the VWE.

From inspection of Eq. (32), it is evident that the size of the step discontinuity at the origin is not a monotonic function of β but instead shows a maximum when $\beta=t_0$, where t_0 is the arrival time of a pressure impulse in a lossless medium [see Eq. (36) below]:

$$t_0 = |x|/c_0. \quad (33)$$

Substituting this value of β into Eq. (32) yields the maximum step size as

$$p_{\max} = \frac{\rho_0 Qc_0}{2t_0}e^{-1}. \quad (34)$$

By normalizing the pressure step in Eq. (32) to the maximum in Eq. (34), the relative step size can be expressed as

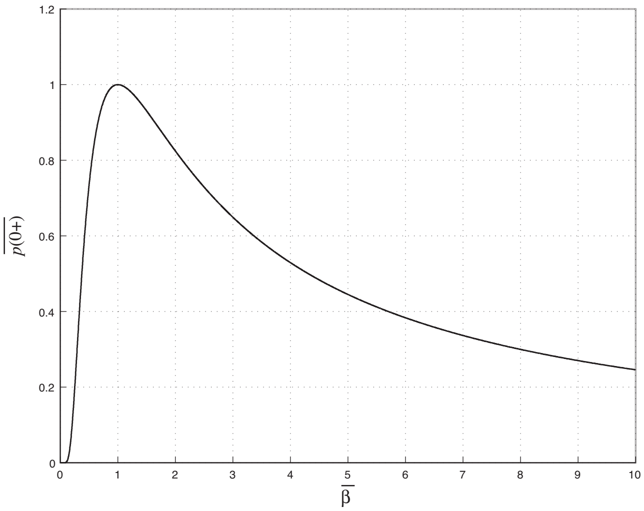

FIG. 1. Height of the VWE step discontinuity at $t=0$ as a function of the normalized bubble-radius coefficient $\bar{\beta}$, evaluated from Eq. (35).

$$\overline{p(0+)} = \frac{p(x,0+)}{p_{\max}} = \frac{1}{\bar{\beta}} \exp\left(1 - \frac{1}{\bar{\beta}}\right), \tag{35}$$

which is plotted in Fig. 1 as a function of the normalized coefficient $\bar{\beta} = \beta / t_0$.

Returning to the Green's function solution in Eq. (30), it is readily shown that, for the case of a lossless medium, represented by $\beta = \gamma = 0$, the integral reduces to a delta function, allowing the pressure to be expressed as

$$p(x,t) = \frac{\rho_0 Q c_0}{2} \delta\left(t - \frac{|x|}{c_0}\right) \quad \text{for } \beta = \gamma = 0. \tag{36}$$

This will be recognized as the correct form for the pressure impulse in an inviscid fluid; it is trivial to show that an identical result is also returned by Eq. (31).

Generally, when β and γ are finite, the integrals in Eqs. (30) and (31) cannot be evaluated explicitly but must be computed numerically. Taking the slightly simpler formulation in Eq. (31), it is convenient to work with the following normalized parameters and variables, denoted by overbars:

$$\bar{\beta} = \frac{\beta}{t_0}; \quad \bar{\gamma} = \frac{\gamma}{t_0}; \quad \bar{t} = \frac{t}{t_0}; \quad \bar{\omega} = \omega t_0; \quad \overline{p(t)} = \frac{p(x,t)}{p_{\max}}. \tag{37}$$

The normalized form of Eq. (31) is then

$$\overline{p(t)} = \frac{e}{2\pi} \int_{-\infty}^{\infty} (1 + i\bar{\omega}\bar{\gamma} - \bar{\beta}^2 \bar{\omega}^2)^{-1/2} \times \exp\left\{ - \frac{i\bar{\omega}}{\sqrt{1 + i\bar{\omega}\bar{\gamma} - \bar{\beta}^2\bar{\omega}^2}} \right\} e^{i\bar{\omega}\bar{t}} d\bar{\omega}. \tag{38}$$

An example of the VWE pressure Green's function, computed from Eq. (38) using a Simpson's rule algorithm, is shown in Fig. 2(a), where the nonphysical discontinuity at the origin is clearly evident. It should be noted that the discontinuity illustrated in Fig. 2(a) is associated exclusively with the bubble-radius term in the VWE. A similar discontinuity will appear in any transient solution of the VWE whenever the source spectrum contains nonzero Fourier components at indefinitely high frequencies.

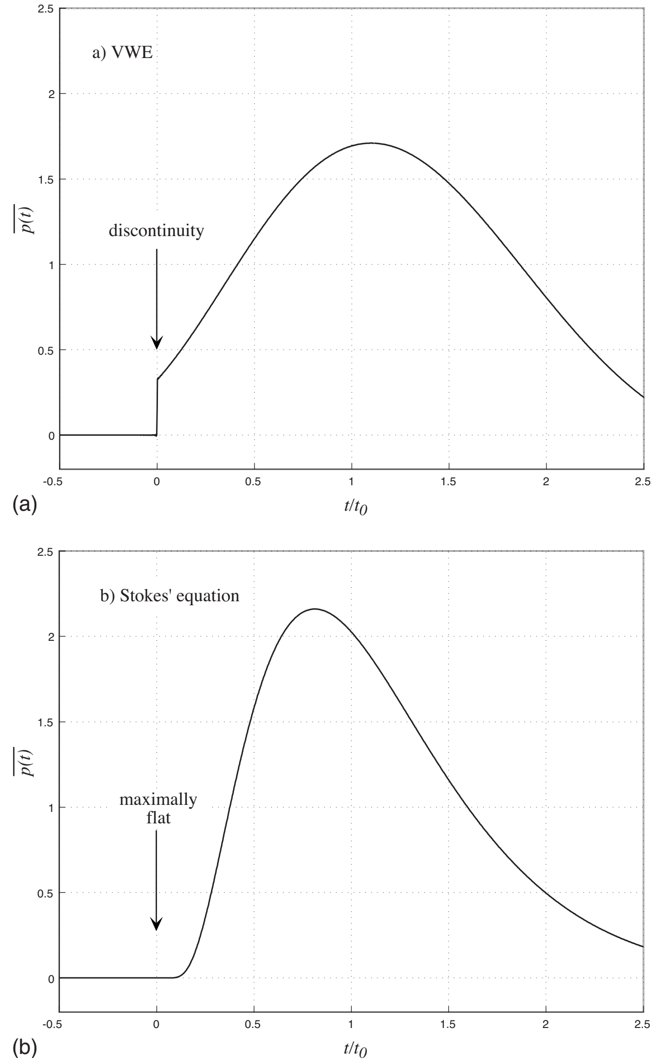

FIG. 2. Dimensionless pressure Green's function, computed from Eq. (38) using the normalization scheme described in the text. (a) The VWE with $\beta = \gamma$. (b) Stokes' equation, $\beta = 0$.

C. An integral identity

The equivalence of the expressions in Eqs. (30) and (31) leads to an explicit expression for the integral in Eq. (31) for the special case $t = 0+$. Under this condition, since Eq. (30) has the value given by Eq. (32), it follows that

$$\int_{-\infty}^{\infty} (1 + i\omega\gamma - \beta^2\omega^2)^{-1/2} \exp\left\{ - \frac{i\omega|x|}{c_0\sqrt{1 + i\omega\gamma - \beta^2\omega^2}} \right\} \times e^{i\omega t} d\omega = \frac{2\pi}{\beta} e^{-|x|/\beta c_0}, \quad \text{for } \beta > 0 \text{ and } t = 0+. \tag{39}$$

Obviously, for $t < 0$ the integral here is zero. Note that, as the integral in Eq. (39) is, in effect, a measure of the step size at the origin of time, it depends on β but not on γ, consistent with the statements made earlier in connection with Eq. (32).

D. Dispersion curves

To obtain the dispersion equations for the VWE, a Fourier transform with respect to time is applied to Eq. (22), which yields

$$\frac{\partial^2 g_\omega}{\partial x^2}+\frac{\omega^2}{c_0^2(1+i\omega\gamma-\beta^2\omega^2)}g_\omega=-Q\frac{\delta(x)}{(1+i\omega\gamma-\beta^2\omega^2)}. \tag{40}$$

The complex sound speed is therefore

$$c(\omega)=c_0\sqrt{1+i\omega\gamma-\beta^2\omega^2}, \tag{41}$$

from which the phase speed is given by

$$\begin{aligned}c_p(\omega)&=\left[\mathrm{Re}\left(\frac{1}{c}\right)\right]^{-1}\\&=\frac{\sqrt{2}c_0\sqrt{(1-\beta^2\omega^2)^2+\omega^2\gamma^2}}{[(1-\beta^2\omega^2)+\sqrt{(1-\beta^2\omega^2)^2+\omega^2\gamma^2}]^{1/2}}\\&\to\begin{cases}c_0 & \text{as } \omega\to 0\\ \dfrac{2c_0\beta^3\omega^2}{\gamma} & \text{as } \omega\to\infty\end{cases}\end{aligned} \tag{42a}$$

and the attenuation is

$$\begin{aligned}\alpha_p(\omega)&=-\frac{|\omega|}{c_0}\mathrm{Im}\left(\frac{1}{c}\right)\\&=\frac{|\omega|}{\sqrt{2}c_0}\frac{[\sqrt{(1-\beta^2\omega^2)^2+\omega^2\gamma^2}-(1-\beta^2\omega^2)]^{1/2}}{\sqrt{(1-\beta^2\omega^2)^2+\omega^2\gamma^2}}\\&\to\begin{cases}\dfrac{\gamma\omega^2}{2c_0} & \text{as } \omega\to 0\\ \dfrac{1}{\sqrt{2}\beta c_0} & \text{as } \omega\to\infty\end{cases}.\end{aligned} \tag{42b}$$

In Fig. 3(a), normalized versions of Eqs. (42) are plotted as functions of the dimensionless frequency $\omega\gamma$. These plots reveal the origin of the step discontinuity in the pressure Green's function at $t=0$. Immediately after the source is activated, the behavior of the Green's function is dominated by its high-frequency Fourier components, which exhibit a phase speed that diverges as the square of the frequency but an attenuation that asymptotes to a constant. Thus, in the limit of high frequency, the VWE predicts Fourier components that travel infinitely fast through the medium while experiencing only finite attenuation. These infinitely fast, propagating waves gives rise to instantaneous arrivals everywhere in the medium and are responsible for the nonphysical step response in the Green's function at the instant the source is activated.

E. Kramers-Kronig relations

The relationship between complex compressibility and complex sound speed is given in Eq. (16). It follows from this expression, in conjunction with Eq. (41), that the complex compressibility for the VWE is

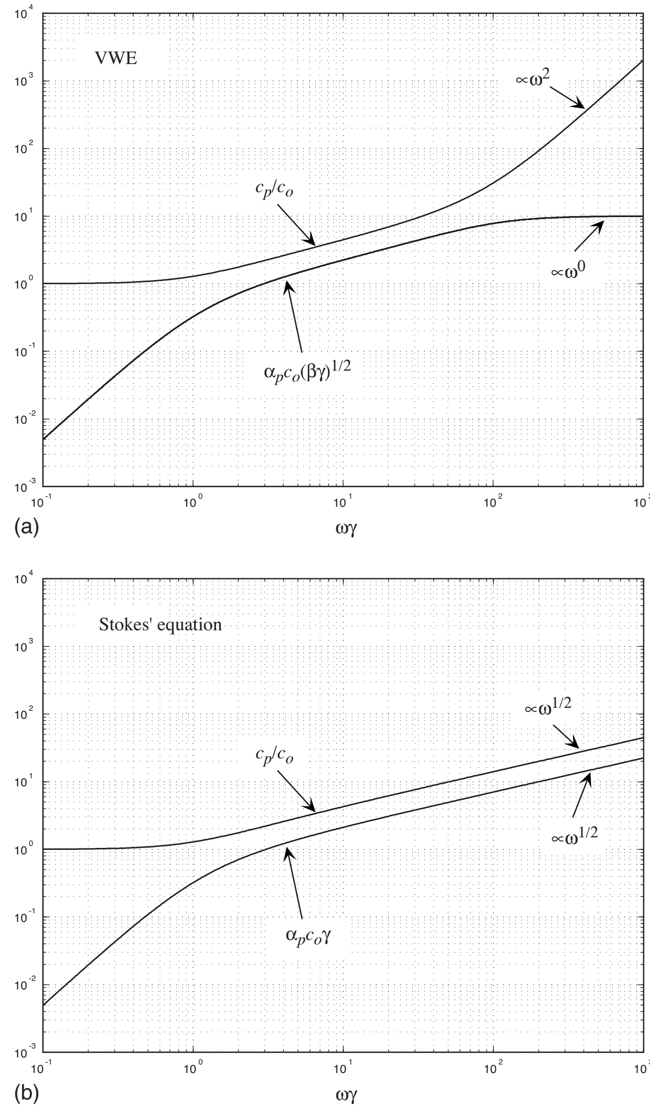

FIG. 3. Normalized dispersion curves as functions of the dimensionless frequency $\omega\gamma$ for (a) the VWE, computed from Eqs. (42a) and (42b) with $\beta=0.1\gamma$ and (b) Stokes' equation, computed from Eqs. (50a) and (50b).

$$K(\omega)=\frac{1}{\rho_0c_0^2(1+i\omega\gamma-\beta^2\omega^2)}, \tag{43a}$$

where the denominator has two roots, $a_\pm$, given by

$$a_\pm=\frac{i\gamma\pm\sqrt{4\beta^2-\gamma^2}}{2\beta^2}. \tag{43b}$$

For $\beta>0$, both of these roots lie in the upper half of the complex ω-plane, irrespective of whether the radical is real or imaginary. Since both poles lie in the upper half-plane, the complex compressibility is analytic on and below the real axis, a corollary of which is that $k(t)$, the inverse Fourier transform of $K(\omega)$, is zero for $t<0$. By applying an inverse Fourier transform to Eq. (43a), the complete formulation of $k(t)$ is obtained:

$$k(t) = u(t)\frac{2}{\rho_0 c_0^2\sqrt{4\beta^2-\gamma^2}}e^{-\gamma t/2\beta^2}\sin\left(\frac{\sqrt{4\beta^2-\gamma^2}}{2\beta^2}t\right). \quad (44)$$

Thus, $k(t)$ is causal, and hence from Titchmarsh's theorem[9,19] the Kramers-Kronig dispersion relations in Eq. (13a) and (13b) must hold for the VWE.

Of course, it could be established that the VWE satisfies Kramers-Kronig by a direct evaluation of the residue expressions in Eqs. (14a) and (14b). To this end, the real and imaginary parts of the complex compressibility are, respectively,

$$K_1(\omega) = \frac{1-\omega^2\beta^2}{\rho_0 c_0^2[(1-\omega^2\beta^2)^2+\omega^2\gamma^2]} \quad (45a)$$

and

$$K_2(\omega) = \frac{\omega\gamma}{\rho_0 c_0^2[(1-\omega^2\beta^2)^2+\omega^2\gamma^2]}, \quad (45b)$$

both of which have poles above and also below the real axis in the complex ω-plane. In the upper half-plane, the two poles are located at $\omega = a_\pm$, where $a_\pm$ are the roots of the denominator of $K(\omega)$, as specified in Eq. (43b); in the lower half-plane, the two poles are at the conjugate positions, $\omega = a_\pm^*$. From the residues of these two conjugate poles, it is straightforward to show that the Kramers-Kronig dispersion relations, as formulated in Eqs. (14a) and (14b), are indeed satisfied by the VWE. Moreover, it is readily verified that the residues from the conjugate poles in the lower half-plane also satisfy the identity in Eq. (15).

IV. STOKES' WAVE EQUATION

Acoustic waves propagating in a viscous fluid are governed by a classical wave equation originally derived by Stokes.[2] For one-dimensional propagation along the x-axis, the inhomogeneous form of Stokes' equation for an impulsive source is

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 g}{\partial t^2} + \gamma\frac{\partial^3 g}{\partial t \partial x^2} = -Q\delta(x)\delta(t), \quad (46)$$

which is identical to the VWE but with the bubble-radius coefficient β^2 set to zero. The absence of the bubble-radius term is responsible for a significant difference in behavior between Stokes' equation and the VWE. Whereas the VWE predicts nonphysical, instantaneous arrivals due to the presence of the bubble-radius term, the transient solutions of Stokes' equation[7] are perfectly well behaved at the instant the source is activated.

A. The pressure Green's function

The Green's function predicted by Stokes' equation may be obtained by setting $\beta=0$ in either of the VWE solutions, Eq. (30) or (31). In these equivalent solutions, there are no arrivals prior to the source being activated, and hence Stokes' equation, like the VWE, satisfies causality. With β set to zero, the normalized Green's function in Eq. (38) is plotted in Fig. 2(b), where it can be seen that no unphysical step discontinuity is present at the instant the source is activated. This is verified by setting $\beta=0$ under the integral in Eq. (32), to obtain the pressure at $t=0+$:

$$p(x,0+) = \frac{\rho_0 Q c_0^2}{2\pi}\int_{-\infty}^{\infty} e^{isx}ds = \rho_0 Q c_0^2 \delta(x), \quad (47)$$

which indicates that, at the instant the source is activated, the pressure is identically zero everywhere in the medium apart from the source position itself. In fact, as discussed in detail by Buckingham,[7] the impulse response of Stokes' equation is maximally flat at the origin in time: not only the pulse itself but all of its time derivatives are zero at $t=0$, giving rise to a perfectly smooth transition from $t=0-$ to $t=0+$. Thus, as well as being causal, the transient solutions of Stokes' equation are physically realizable, implying a finite speed for all the propagating Fourier components and no instantaneous arrivals.

B. Two integral identities

The equivalence of the expressions in Eqs. (30) and (31) leads to an explicit expression for the integral in Eq. (31) for the special case when $\beta=0$ and $t=0+$. Under this condition, since Eq. (30) has the value given by Eq. (47), it follows that

$$\int_{-\infty}^{\infty}(1+i\omega\gamma)^{-1/2}\exp\left\{-\frac{i\omega|x|}{c_0\sqrt{1+i\omega\gamma}}\right\}e^{i\omega t}d\omega = 2\pi c_0\delta(x) \quad \text{for } t=0+. \quad (48)$$

Thus, the integral on the left is zero for all $|x|\neq 0$ when $t=0+$. The same integral is zero everywhere for $t\leqslant 0$, that is

$$\int_{-\infty}^{\infty}(1+i\omega\gamma)^{-1/2}\exp\left\{-\frac{i\omega|x|}{c_0\sqrt{1+i\omega\gamma}}\right\}e^{-i\omega t}d\omega = 0 \quad \text{for } |x|>0 \text{ and } t\geqslant 0. \quad (49)$$

C. Dispersion curves

The dispersion curves associated with Stokes' equation may be obtained directly from Eqs. (42a) and (42b) by setting $\beta=0$, under which condition the phase speed is

$$c_p(\omega) = \frac{\sqrt{2}c_0\sqrt{1+\omega^2\gamma^2}}{[1+\sqrt{1+\omega^2\gamma^2}]^{1/2}} \to \begin{cases} c_0 & \text{as } \omega\to 0 \\ c_0\sqrt{2\omega\gamma} & \text{as } \omega\to\infty \end{cases} \quad (50a)$$

and the attenuation is

$$\alpha_p(\omega) = \frac{|\omega|}{\sqrt{2}c_0}\frac{[\sqrt{1+\omega^2\gamma^2}-1]^{1/2}}{\sqrt{1+\omega^2\gamma^2}} \to \begin{cases} \dfrac{\gamma\omega^2}{2c_0} & \text{as } \omega\to 0 \\ \dfrac{1}{c_0}\sqrt{\dfrac{\omega}{2\gamma}} & \text{as } \omega\to\infty \end{cases}. \quad (50b)$$

Normalized versions of Eqs. (50a) and (50b) are plotted in Fig. 3(b) as functions of the dimensionless frequency $\omega\gamma$.

An important distinction between the VWE and Stokes' equation is illustrated in Fig. 3. At high frequencies, the VWE predicts a phase speed that increases as the square of the frequency and an attenuation that asymptotes to a finite, constant value, whereas Stokes' equation yields a phase speed and attenuation that both scale as the square-root of frequency. Thus, in the limit of high frequency, both equations predict infinitely fast Fourier components, which are infinitely attenuated in the case of Stokes' equation, but propagate finite distances according to the VWE. These infinitely fast, propagating waves are responsible for the unphysical discontinuity predicted by the VWE at the instant the source is activated. In contrast, the infinitely fast waves predicted by Stokes' equation do not propagate through the medium because they have zero amplitude, due to the infinite attenuation, and accordingly the solution is well behaved at the time the source is activated, showing no step discontinuity at $t=0$.

D. Kramers-Kronig

The real and imaginary parts of the complex compressibility from Stokes' equation are given by Eqs. (45a) and (42b) with β set to zero:

$$K_1(\omega) = \frac{1}{\rho_0 c_0^2(1+\omega^2\gamma^2)} \tag{51a}$$

and

$$K_2(\omega) = \frac{\omega\gamma}{\rho_0 c_0^2(1+\omega^2\gamma^2)}. \tag{51b}$$

Both of these functions have two simple poles symmetrically placed on the imaginary axis at $\omega = \pm i/\gamma$. The negative pole, of course, lies in the lower half-plane and therefore falls within the contour of integration used to derive the dispersion relations in Eqs. (14a) and (14b). It follows that, in both Eqs. (14a) and (14b), the residue of this pole is nonzero, thus making a finite contribution to the corresponding integrals in Eqs. (13a) and (13b).

It is almost trivial to verify that Eqs. (51a) and (51b) satisfy the Kramers-Kronig dispersion relationships, as expressed in Eqs. (14a) and (14b). Alternatively, the dispersion integrals in Eqs. (13a) and (13b) may be evaluated directly by factorizing the integrands and then making an elementary substitution, bearing in mind that the integral of an odd function taken over infinite limits is zero. Either way, it is found that Eqs. (51a) and (51b) do indeed satisfy Kramers-Kronig. Of course, this is only to be expected, since $K(\omega)$ has no poles on or below the real axis, hence $k(t)$ is causal and, from Titchmarsh's theorem, it follows that the Kramers-Kronig relations must be satisfied.

It is a straightforward matter to derive the full expression for the Stokes' equation compressibility $k(t)$, by taking the limit of Eq. (44) as β approaches zero. This procedure yields the expression

$$k(t) = u(t)\frac{1}{\rho_0 c_0^2 \gamma}e^{-t/\gamma}. \tag{52}$$

V. THE TIME-DEPENDENT DIFFUSION EQUATION (TDDE)

A commonly encountered partial differential equation of theoretical physics is the time-dependent diffusion equation (TDDE), the one-dimensional, inhomogeneous form of which is

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{c_\infty^2}\frac{\partial^2 g}{\partial t^2} - \eta\frac{\partial g}{\partial t} = -Q\,\delta(x)\,\delta(t), \tag{53}$$

where η is a positive constant and c_∞ is the speed of sound in the medium in the limit of high frequency. Buckingham,[22] in the context of acoustic propagation through a porous medium with a rigid frame, developed the solution of Eq. (53) by applying two Fourier transforms, one temporal and the other spatial, to obtain an algebraic equation for the doubly transformed field. The corresponding inverse Fourier transforms were then applied, yielding the impulse response for the velocity potential:

$$g(x,t) = \frac{Qc_\infty}{2}u\left(t - \frac{|x|}{c_\infty}\right)\exp\left(-\frac{\eta c_\infty^2}{2}t\right)I_0\left(\frac{\eta c_\infty}{2}\sqrt{c_\infty^2 t^2 - x^2}\right), \tag{54}$$

where $I_0(..)$ is the modified Bessel function of the first kind of order zero. An equivalent solution can also be found in Morse and Feshbach,[23] derived in connection with the diffusion of heat in a gas.

It is clear from the presence of the step function that the solution in Eq. (54) is causal and physically realizable, since all the arrivals occur after a finite time: $t=t_\infty=|x|/c_\infty$. Unlike the case of the VWE, no instantaneous arrivals are predicted. As illustrated in Fig. 4, the onset of the pulse is abrupt, in contrast with the perfectly smooth impulse response predicted by Stokes' equation.

A Fourier transform of Eq. (53) with respect to time shows that the complex sound speed from the TDEE is given by

$$c(\omega) = \left(\frac{1}{c_\infty^2} - i\frac{\eta}{\omega}\right)^{-1/2}. \tag{55}$$

It follows that the phase speed is

$$c_p(\omega) = \sqrt{2}c_\infty\left[1 + \sqrt{1 + \frac{\eta^2 c_\infty^4}{\omega^2}}\right]^{-1/2} \to \begin{cases} \sqrt{\dfrac{2\omega}{\eta}} & \text{for } \omega \ll \eta c_\infty^2 \\ c_\infty & \text{for } \omega \gg \eta c_\infty^2 \end{cases} \tag{56a}$$

and the attenuation is

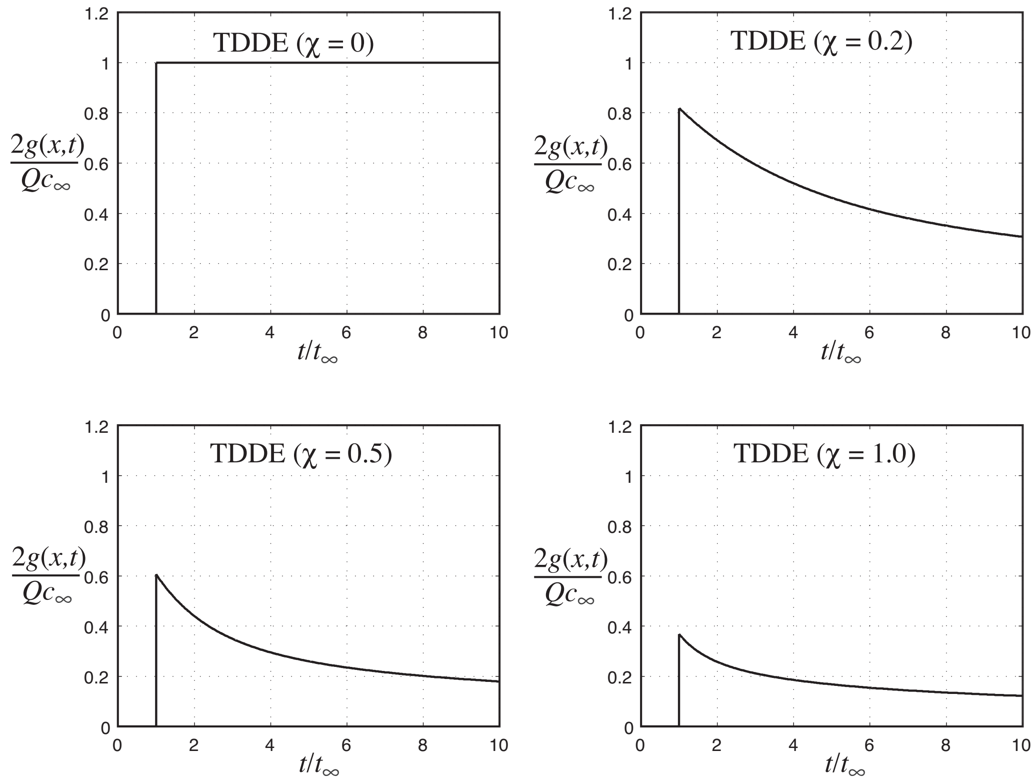

FIG. 4. Dimensionless velocity potential predicted by the TDDE, computed from Eq. (54), for four values of the parameter $\chi = \eta c_\infty |x|/2$.

$$\alpha_p(\omega) = \frac{\omega}{\sqrt{2}c_\infty}\left[\sqrt{1+\frac{\eta^2 c_\infty^4}{\omega^2}}-1\right]^{1/2} \rightarrow \begin{cases} \sqrt{\dfrac{\eta\omega}{2}} & \text{for } \omega \ll \eta c_\infty^2 \\ \dfrac{\eta c_\infty}{2} & \text{for } \omega \gg \eta c_\infty^2 \end{cases}. \tag{56b}$$

These dispersion equations are plotted in dimensionless form in Fig. 5, where it can be seen that the wave speed and attenuation are both finite in the limit of high frequency. Such behavior is consistent with the absence of instantaneous arrivals in the Green's function in Eq. (54).

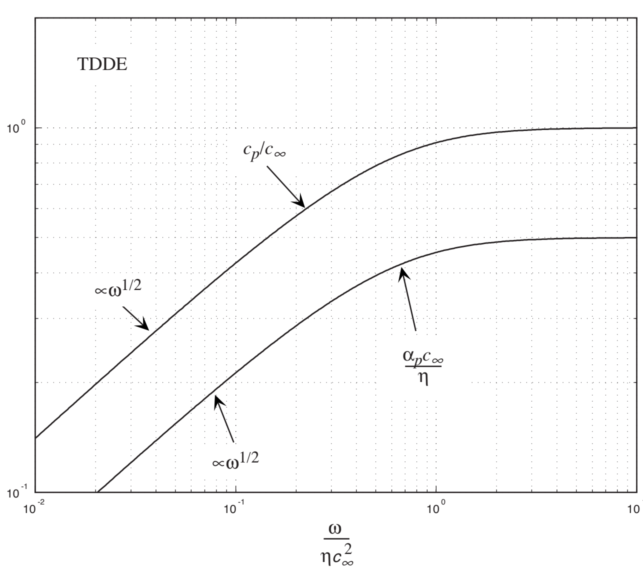

FIG. 5. Normalized dispersion curves for the TDDE, computed from Eqs. (56a) and (56b), as functions of the dimensionless frequency $\omega/\eta c_\infty^2$.

The complex compressibility is obtained directly from Eq. (55):

$$K(\omega) = \frac{1}{\rho_0 c^2(\omega)} = \frac{1}{\rho_0 c_\infty^2} - i\frac{\eta}{\rho_0\omega}, \tag{57}$$

from which

$$K_1(\omega) = \frac{1}{\rho_0 c_\infty^2} \tag{58a}$$

and

$$K_2(\omega) = \frac{\eta}{\rho_0\omega}. \tag{58b}$$

When substituted into the integrals in Eqs. (13a) and (13b), it is evident that these expressions fail to satisfy the Kramers-Kronig dispersion relations. In fact, both integrals return a value of zero for $\omega \neq 0$. The failure could, perhaps, have been anticipated, since the convergence condition in Eq. (8), which underpins the Kramers-Kronig relations, is not satisfied by the complex compressibility in Eqs. (56a) and (56b): the real part of $K(\omega)$ is independent of frequency and hence remains finite in the limit of high frequency.

A similar problem occurs with optical dispersion in a dielectric but is overcome by working with a function that is equivalent to $U(\omega)=K(\omega)-K_1(\infty)$, which, in the electromagnetic case, does satisfy the convergence condition in Eq. (8). Such a procedure does not solve the problem with the TDDE, however, even though $U(\omega)$ satisfies Eq. (8), as may

TABLE I. Causal properties of wave equations.

	Causal impulse response function?	Physically realizable impulse response function?	Kramers-Kronig satisfied?
van Wijngaarden's equation	Yes	No	Yes
Stokes' equation	Yes	Yes	Yes
Time-dependent diffusion equation	Yes	Yes	No

be appreciated from the fact that the real and imaginary parts of $U(\omega)$ fail to satisfy the Kramers-Kronig dispersion relation in Eq. (13b).

The fundamental difficulty with the TDDE is that neither $K(\omega)$ nor $U(\omega)$ is square-integrable and therefore the corresponding inverse Fourier transforms do not exist. But the causality argument that is central to the derivation of the Kramers-Kronig dispersion relations is predicated upon the existence of $k(t)$. Thus, the TDDE is an example of a wave equation that does not satisfy Kramers-Kronig but whose transient solutions are causal and physically realizable.

VI. CONCLUDING REMARKS

The impulse-response solutions of three wave equations, van Wijngaarden's equation (VWE), Stokes' equation, and the time-dependent diffusion equation (TDDE), have been developed in this article, and the causal properties of the equations themselves and their solutions examined in some depth. These properties are summarized in Table I, where it can be seen that all three equations return impulse-response functions that are strictly causal; that is, no acoustic arrivals are predicted anywhere in the medium prior to the activation of the source.

Stokes' equation for the propagation of acoustic waves in a viscous fluid is perfectly well behaved in that the impulse response is physically realizable, with no instantaneous arrivals predicted anywhere in the medium at a finite distance from the source; the complex compressibility satisfies the Kramers-Kronig dispersion relations. At first sight, the absence of instantaneous arrivals may be surprising since, at high frequencies, the phase speed increases without limit, scaling as frequency to the power of one-half. However, the attenuation follows the same scaling law, also increasing without limit as the square-root of frequency (Fig. 3(b)). Thus, the infinitely fast Fourier components are infinitely attenuated and hence do not propagate through the medium. In effect, because they have zero amplitude, such components make no contribution to the solution of Stokes' equation. Consequently, the impulse response is absolutely physical, and exhibits a particularly interesting feature at $t=0$, the time at which the source is activated: the response is maximally flat when $t=0$, that is, the impulse response itself and all its time derivatives are zero, giving rise to a perfectly smooth transition through the origin of time.

Taken in isolation, the fact that Stokes' equation predicts wave speeds that increase without limit could conceivably be construed as unphysical, but this would be a misinterpretation. The transient solutions of a wave equation are intimately connected to the frequency dependence of not just the wave speed but also the attenuation. A wave equation is physical provided it is causal and predicts no instantaneous arrivals, criteria which are both satisfied by Stokes' equation.

The same cannot be said of Van Wijngaarden's equation for the propagation of acoustic waves through a viscous, bubbly liquid. Although the VWE is causal and satisfies Kramers-Kronig, its impulse-response function is not physically realizable in that it predicts instantaneous acoustic arrivals everywhere in the medium. These nonphysical contributions to the solution are due solely to the presence of the bubble-radius term with nonzero coefficient β^2 in Eq. (22). This term has two effects on the high-frequency behavior of the dispersion curves [Fig. 3(a)]: above the transition frequency $\omega=\beta^{-1}$, the phase speed diverges as the square of frequency and the attenuation asymptotes to a constant, finite value. Thus, the bubble-radius term introduces indefinitely fast, finite-amplitude Fourier components into the solution for the impulse response, and these components propagate finite distances through the medium, arriving everywhere at the same instant that the source is activated. Such a phenomenon is physically impossible in any material that is subject to the constraints of continuum mechanics.

The time-dependent diffusion equation predicts an impulse-response function that is physically realizable, with no instantaneous arrivals, but its complex compressibility does not satisfy the Kramers-Kronig dispersion relations. This failure is interesting because Kramers-Kronig is generally taken as a definitive test of causality. So, here is a situation where a wave equation possesses a causal, physically realizable impulse response, yet, judged on the basis of Kramers-Kronig, would be said to be noncausal. The origin of the contradiction lies with the Kramers-Kronig relations themselves. They are predicated upon the causal behavior of the compressibility $k(t)$, which is the inverse Fourier transform of the complex compressibility $K(\omega)$. But, in this case, $K(\omega)$ is not square-integrable, therefore $k(t)$ does not exist (mathematically), and hence the argument leading to the Kramers-Kronig relations fails.

A few final comments are perhaps in order. An impulsive source, as considered in this article, has an infinite bandwidth and is thus an idealized representation of any actual physical source, which is always band-limited. Now, the solution for the acoustic field produced by an impulsive source is the Green's function $g(t)$. If a finite-bandwidth source had been considered, with pulse shape $f(t)$, then it is readily shown that the solution for the time-dependent field is the convolution $f(t)\otimes g(t)$, which, in the frequency domain, becomes the product $F(\omega)G(\omega)$, where the upper case letters are the Fourier transforms of their lower case counterparts.

Since $F(\omega)$ is zero beyond some upper-frequency limit, the solution for the field due to the band-limited source does not contain indefinitely high frequencies. Although $f(t)$ will affect shapes of the transient solutions of Stokes' equation and the TDDE, the band-limited source introduces no fundamentally new behavior into these solutions. In the case of the VWE, however, the absence of indefinitely high frequencies in the source function means that instantaneous arrivals do not appear in the solution for the band-limited transient field. In fact, if the source bandwidth were reduced below β^{-1}, the bubble-radius term would, in effect, become neutralized and the solution of the VWE would reduce to that of Stokes' equation.

All of which leads to the question: Is the VWE a valid descriptor of acoustic propagation in a viscous, bubbly liquid or, indeed, in any physically realizable medium? Arguably, the VWE is invalid because, in the limit of high frequency, it returns nonphysical solutions. Just because infinitely high frequencies cannot be produced in practice does not alter the fact that the predicted field must show the correct limiting behavior, which, of course, the VWE fails to do. The situation is much like that encountered with a radiation condition, where the field must go to zero infinitely far from the source. The fact that a receiver could never be placed at an infinite distance from the source does not invalidate the requirement that the solution for the field must satisfy the boundary condition at infinity. Similarly, any wave equation which admits a solution that is nonphysical in any part of the spectrum, including the region extending to indefinitely high frequencies, must be regarded as flawed. In the case of the VWE, the instantaneous arrivals are attributable directly to the presence of the bubble-radius term, suggesting either that this term is incorrect or that another term needs to be included which would modify the high-frequency response. Be that as it may, the instantaneous arrivals represent a deficiency of the VWE which, accordingly, should be treated with caution.

ACKNOWLEDGMENTS

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